Sample Problems

1. Find the value of m so that the quadratic equation $x^2 - 8x + 6m = 2$ has exactly one solution for x.
2. Consider the equation $x^2 + 2mx + 46 = 10x + 6m$. Find all values of m for which the equation has exactly one real solution for x.
3. Find an equation of the tangent line drawn to the graph of $y = x^2 - 9x + 7$ with slope -3.
4. Find an equation of the tangent line(s) drawn to the graph of $y = \frac{1}{2}x^2 + 3x + 3$ from the point $(0, 1)$.
5. Find an equation for all tangent lines drawn to the graph of $f(x) = \frac{1}{2}x^2 - 6x + 30$ from the point $(7, 8)$.
6. Find an equation for all tangent lines drawn to the graph of $y = x^2 - 10x + 16$ from the point $(7, -14)$.

Practice Problems

1. Find the value of p so that the quadratic equation $x^2 + 10x + 3p - 8 = 0$ has exactly one solution for x.
2. Consider the parametric equation $x^2 - 6x + 49 = 2ax + 12a$. Find all values of a for which the equation has one real solution for x.
3. Find an equation of the tangent line drawn to the graph of $y = x^2 - 17x + 59$ with slope -3.
4. In each case, find an equation for all tangent lines drawn to the given graph from the point given.

   a) $y = x^2 - 8x + 3$ from $P(0, -6)$

   b) $f(x) = 7x - \frac{1}{2}x^2 - 20$ from $P(4, 8)$

   c) $y = \frac{1}{2}x^2 + 3x - 11$ from $P(2, -5)$

   d) $f(x) = \frac{1}{2}x^2 - x + 2$ from $P(3, -1)$

Sample Problems - Answers

1. 3 2. $7, -3$ 3. $y = -3x - 2$ 4. $y = x + 1$ and $y = 5x + 1$ 5. $y = 4x - 20$ and $y = -2x + 22$
6. $y = -2x$ and $y = 10x - 84$

Practice Problems – Answers

1. 11 2. $-20,\ 2$ 3. $y = -3x + 10$

4. a) $y = -2x - 6$ and $y = -14x - 6$ b) $y = -x + 12$ and $y = 7x - 20$ c) $y = 3x - 11, y = 7x - 19$
d) $y = -x + 2$ and $y = 5x - 16$

Sample Problems – Solutions

1. Find the value of m so that the quadratic equation $x^2 - 8x + 6m = 2$ has exactly one solution for x.

   Solution: This equation is quadratic in x. To solve, we reduce one side to zero and write the quadratic expression as a polynomial in x, where each coefficient is a polynomial in m.

$$\begin{aligned} x^2 - 8x + 6m &= 2 \\ x^2 - 8x + 6m - 2 &= 0 \end{aligned}$$

   Recall that the quadratic equation $ax^2 + bx + c = 0$ with $a \neq 0$ has exactly one real solution if its discriminant, $b^2 - 4ac$ is zero. The coefficients of this equation are

$$a = 1, \ \ b = -8, \quad \text{and} \ \ c = 6m - 2$$

   We will obtain an equation in m by stating that the discriminant is zero.

$$\begin{aligned} b^2 - 4ac &= 0 \\ (-8)^2 - 4(1)(6m - 2) &= 0 \end{aligned}$$

   This is a linear equation that we can easily solve for m.

$$\begin{aligned} 64 - 24m + 8 &= 0 \\ 72 &= 24m \\ 3 &= m \end{aligned}$$

2. Consider the equation $x^2 + 2mx + 46 = 10x + 6m$. Find all values of m for which the equation has exactly one real solution for x.

   Solution: We reduce one side to zero and express the other side as a polynomial in x., where each coefficient is expressed as a polynomial in m.

$$\begin{aligned} x^2 + 2mx + 46 &= 10x + 6m \\ x^2 + 2mx - 10x + 46 - 6m &= 0 \\ x^2 + (2m - 10)x + (-6m + 46) &= 0 \end{aligned}$$

This equation has coefficients $a = 1$, $b = 2m - 10$, and $c = -6m + 46$. We will obtain an equation in m by stating that the discriminant is zero.

$$\begin{aligned} b^2 - 4ac &= 0 \\ (2m-10)^2 - 4(1)(-6m+46) &= 0 \end{aligned}$$

We now solve the quadratic equation for m.

$$\begin{aligned} (2m-10)^2 - 4(1)(-6m+46) &= 0 \quad \text{divide by } 4 \\ (m-5)^2 - (-6m+46) &= 0 \\ m^2 - 10m + 25 + 6m - 46 &= 0 \\ m^2 - 4m - 21 &= 0 \\ (m-7)(m+3) &= 0 \quad \Longrightarrow \quad m_1 = 7 \qquad m_2 = -3 \end{aligned}$$

Thus the equaton will have one solution when m is 7 or -3.
We can check: if $m = 7$, then

$$\begin{aligned} x^2 + 2mx + 46 &= 10x + 6m \quad \text{and } m = 7 \\ x^2 + 2(7)x + 46 &= 10x + 6(7) \\ x^2 + 14x + 46 &= 10x + 42 \\ x^2 + 4x + 4 &= 0 \\ (x+2)^2 &= 0 \quad \Longrightarrow \quad x = -2 \end{aligned}$$

and if $m = -3$, then

$$\begin{aligned} x^2 + 2mx + 46 &= 10x + 6m \quad \text{and } m = -3 \\ x^2 + 2(-3)x + 46 &= 10x + 6(-3) \\ x^2 - 6x + 46 &= 10x - 18 \\ x^2 - 16x + 64 &= 0 \\ (x-8)^2 &= 0 \quad \Longrightarrow \quad x = 8 \end{aligned}$$

3. Find an equation of the tangent line drawn to the graph of $y = x^2 - 9x + 7$ with slope -3.
Solution: We are looking for the equation of a line with slope -3. Its slope-intercept form is then $y = -3x + p$. We need to find the value of p. The coordinates of all intersection points of the line $y = -3x + p$ and the parabola $y = x^2 - 9x + 7$ are solutions of the system

$$\begin{cases} y = -3x + p \\ y = x^2 - 9x + 7 \end{cases}$$

If we were to solve this solution, we would use substitution and obtained the equation

$$\begin{aligned} x^2 - 9x + 7 &= -3x + p \\ x^2 - 6x + 7 - p &= 0 \end{aligned}$$

The coefficients in this equation are $a = 1$, $b = -6$, and $c = -p + 7$. A quadratic equation has exactly one solution if the discriminant is zero. By stating this, we will obtain an equation in p.

$$\begin{aligned} b^2 - 4ac &= 0 \\ (-6)^2 - 4(1)(-p+7) &= 0 \\ 36 + 4p - 28 &= 0 \\ 4p &= -8 \\ p &= -2 \end{aligned}$$

Thus the tangent line is $y = -3x - 2$.

4. Find an equation of the tangent line(s) drawn to the graph of $y = \frac{1}{2}x^2 + 3x + 3$ from the point $(0, 1)$.
Solution: We are looking for the equation of a line. Specifically, we will find its slope-intercept form, $y = mx + k$. Thus, we need to find the values of m and k. Clearly, $k = 1$ since the point given is the $y-$intercept.

$$\begin{aligned} y &= mx + k \qquad (0,1) \text{ is on the line} \\ 1 &= m(0) + k \\ 1 &= k \qquad \Longrightarrow \quad y = mx + 1 \end{aligned}$$

Thus our tangent line is $y = mx + 1$. This equation represents all non-vertical lines passing through $(0, 1)$. We will obtain the value(s) of m using the fact that it must be a tangent line. The common points of the line and the parabola have coordinates that are solution of the system

$$\begin{cases} y = mx + 1 \\ y = \frac{1}{2}x^2 + 3x + 3 \end{cases}$$

If we use substitution to solve the system, we obtain the equation

$$mx + 1 = \frac{1}{2}x^2 + 3x + 3$$

Once we solve this quadratic equation, we will have the $x-$coordinates of all intersection points. Our line is a tangent line if this quadratic equation has exactly one solution.

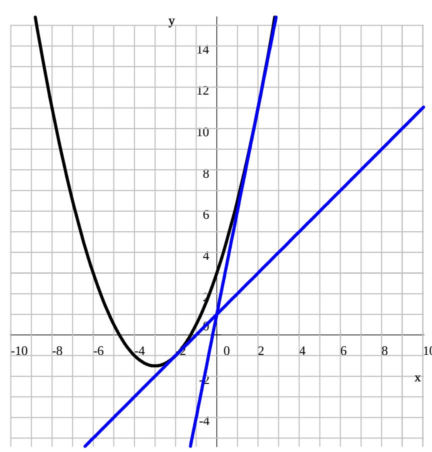
$$\begin{aligned} mx + 1 &= \frac{1}{2}x^2 + 3x + 3 \\ 0 &= \frac{1}{2}x^2 - mx + 3x + 2 \\ 0 &= \frac{1}{2}x^2 + (-m + 3)x + 2 \end{aligned}$$

$$a = \frac{1}{2}, \quad b = -m + 3, \quad \text{and } c = 2$$

A quadratic equation has ecatly one solution if the discriminant is zero. This will give us an equation for m.

$$\begin{aligned} b^2 - 4ac &= 0 \\ (-m+3)^2 - 4\left(\frac{1}{2}\right)2 &= 0 \\ (m-3)^2 - 4 &= 0 \qquad \text{factor via difference of squares theorem} \\ (m - 3 + 2)(m - 3 - 2) &= 0 \\ (m-1)(m-5) &= 0 \quad \Longrightarrow \quad m_1 = 1 \quad m_2 = 5 \end{aligned}$$

Thus the two tangent lines drawn from $(0, 1)$ to the graph of $y = \frac{1}{2}x^2 + 3x + 3$ are $y = x + 1$ and $y = 5x + 1$.



5. Find an equation for all tangent lines drawn to the graph of $f(x) = \frac{1}{2}x^2 - 6x + 30$ from the point $(7, 8)$.

Solution: We start with the equation of all (except vertical) lines passing through $(7, 8)$. These lines can all be expressed using the slope-point form and then solving for y.

$$y - 8 = m(x - 7) \qquad \Longrightarrow \quad y = mx - 7m + 8$$

Now the points of intersection between the parabola and line can be found by solving the system

$$\begin{cases} y = mx - 7m + 8 \\ y = \frac{1}{2}x^2 - 6x + 30 \end{cases}$$

We want to find m so that this system has exactly one solution. We 'pretend' that we know the value of m and compute for the points of intersection. Using substitution, we obtain the following quadratic equation.

$$\frac{1}{2}x^2 - 6x + 30 = mx - 7m + 8$$

We reduce one side to zero and arrange the terms as a polynomial in x, where each coefficient is a polynomial in m.

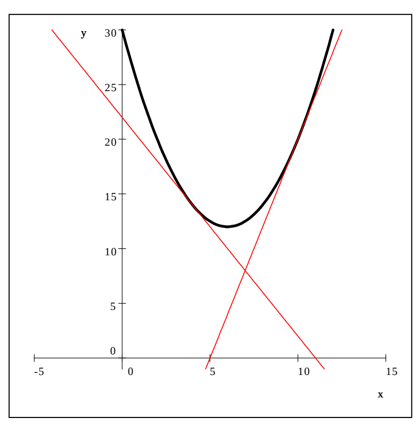
$$\begin{aligned} \frac{1}{2}x^2 - 6x + 30 &= mx - 7m + 8 \\ \frac{1}{2}x^2 - 6x + 30 - mx + 7m - 8 &= 0 \\ \frac{1}{2}x^2 + (-m - 6)x + 7m + 22 &= 0 \end{aligned}$$

$$a = \frac{1}{2}, \quad b = (-m - 6), \quad \text{and} \quad c = 7m + 22$$

A quadratic equation has exactly one solution if its discriminant is zero.

$$\begin{aligned} b^2 - 4ac &= 0 \\ (-m - 6)^2 - 4\left(\frac{1}{2}\right)(7m + 22) &= 0 \\ (-m - 6)^2 - 2(7m + 22) &= 0 \\ m^2 + 12m + 36 - 14m - 44 &= 0 \\ m^2 - 2m - 8 &= 0 \\ (m - 4)(m + 2) &= 0 \quad \Longrightarrow \quad m_1 = 4 \quad m_2 = -2 \end{aligned}$$

Thus the two tangent lines are $y - 8 = 4(x - 7)$ and $y - 8 = -2(x - 7)$. We can simplify these equations to get the slope intercept forms, $y = 4x - 20$ and $y = -2x + 22$.



6. Find an equation for all tangent lines drawn to the graph of $y = x^2 - 10x + 16$ from the point $(7, -14)$.
Solution: We start with the equation of all (except vertical) lines passing through $(7, -14)$. These lines can all be expressed using the slope-point form and then solving for y.

$$y + 14 = m(x - 7) \quad \Longrightarrow \quad y = mx - 7m - 14$$

Now the points of intersection between the parabola and line can be found by solving the system

$$\begin{cases} y = mx - 7m - 14 \\ y = x^2 - 10x + 16 \end{cases}$$

We want to find m so that the system has exactly one solution. The $x-$coordinate of all points of intersection is a solution of the equation

$$\begin{aligned} x^2 - 10x + 16 &= mx - 7m - 14 \\ x^2 - 10x + 16 - mx + 7m + 14 &= 0 \\ x^2 - 10x - mx + 7m + 30 &= 0 \\ x^2 + (-m - 10)x + 7m + 30 &= 0 \quad \Longrightarrow \quad a = 1,\ b = -m - 10,\ \ c = 7m + 30 \end{aligned}$$

A quadratic equation has exactly one real solution if its discriminant is zero.

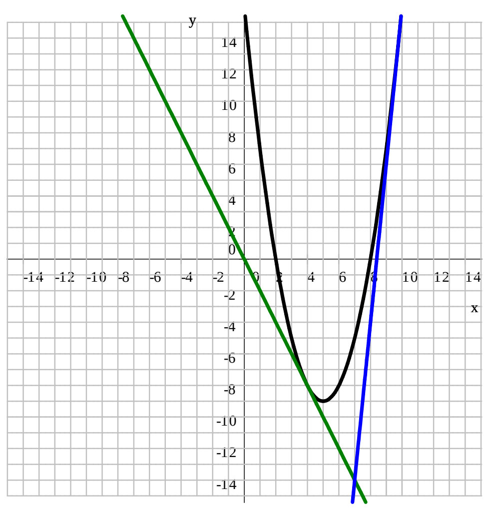
$$\begin{aligned} b^2 - 4ac &= 0 \\ (-m - 10)^2 - 4(1)(7m + 30) &= 0 \\ m^2 + 20m + 100 - 28m - 120 &= 0 \\ m^2 - 8m - 20 &= 0 \\ (m + 2)(m - 10) &= 0 \quad \Longrightarrow \quad m_1 = -2 \qquad m_2 = 10 \end{aligned}$$

We substitute these values into the equaltion of the line $y + 14 = m(x - 7)$ to get the two equations

$$y + 14 = -2(x - 7) \quad \text{and} \quad y + 14 = 10(x - 7)$$

We can bring these to the slope intercept form and get

$$y = -2x \ \text{ (green graph) } \ \text{and} \ \ y = 10x - 84 \ \text{ (blue graph)}$$



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