Definition: A (finite or infinite) list of objects is called a **sequence**. The terms or elements of a sequence are often denoted by a_1, a_2, ...,a_n. The integer k in a_k is called the index of the term a_k.

For the most part, we will study sequences of numbers. We will start with the simpler types.

Definition: An **arithmetic sequence** is a sequence of numbers in which the consecutive terms are increasing or decreasing by the same amount. This same amount is called the common difference of the sequence.

For example, the sequence $10,\ 13,\ 16,\ 19,\ 22,\ \ldots$ can be given by its first term, $a_1 = 10$ and common difference $d = 3$.

We will often denote the first term by a instead of a_1 and express things in terms of a and d

	a_1	a_2	a_3	a_4	$\ldots$
in this case:	13	16	19	22	$\ldots$
	$10+3$	$10+2\cdot 3$	$10+3\cdot 3$	$10+4\cdot 3$	

in general:	a_1	a_2	a_3	a_4	a_5	$\ldots$
	a	$a+d$	$a+2d$	$a+3d$	$a+4d$	$\ldots$

This is however, not always the most useful notation, as we will see later.

There are usually two questions often asked about arithmetic sequences. Given the sequence, what is its nth term, and what is the sum of its first n terms.

To find the nth term, we can see that it is just a matter of adding the common difference repeatedly to the first term. Not exactly n times for a_n, but almost. If we inspect the first few terms, we see the pattern.

$$\begin{aligned} a_1 &= a \\ a_2 &= a+d \\ a_3 &= a+2d \\ a_4 &= a+3d \\ &\vdots \\ a_n &= a+(n-1)\,d \end{aligned}$$

We do not get the nth term by adding the common difference to the first term exactly n-times. This might remind us of a fact from counting. To count out 100 numbers, we can count from 1 to 100, or from 0 to 99. This is the case of counting from zero, as $a_1 = a = a + 0 \cdot d$

If $a_1, a_2, \ldots$ is an arithmetic sequence with first term a and common difference d, then the nth term, a_n can be computed as

$$a_n = a + (n-1)\,d$$

Example 1. Consider the arithmetic sequence with first term -500 and common difference 7.

a) Find the 200th term in the sequence. b) Find k if $a_k = 25$.

Solution: a) We apply the formula with $a = -500$, $d = 7$, and $n = 200$.

$$a_{200} = a + 199d = -500 + 199 \cdot 7 = 893$$

b) We apply the same formula and solve for k.

$$\begin{aligned} 25 &= -500 + (n-1)\,7 \\ 525 &= 7\,(n-1) \\ 75 &= n-1 \\ 76 &= n \end{aligned}$$

Therefore, 25 is the 76th term of this sequence, We can also find the sum of the first n terms, often denoted by s_n.

Carl Friedrich Gauss (1777-1855) is probably one of the greatest mathematicians of all time. He made major contributions to most areas within mathematics. The following story is from when Gauss was still at primary school. One day Gauss' teacher asked his class to add together all the numbers from 1 to 100, assuming that this task would occupy them for quite a while. He was shocked when young Gauss, after a few seconds thought, wrote down the answer 5050. The teacher couldn't understand how his pupil had calculated the sum so quickly in his head, but the eight year old Gauss pointed out that the problem was actually quite simple.

Here is the question: $1+2+3+...+100 = x$

We repeat this line, but this time, backward: $100+99+98+...+1 = x$

And we will add the two lines - the result will be twice the desired sum. Instead of adding row 1 and then row 2, we will add the numbers column by column. The sum in the first column is $1+100 = 101$.

$$\begin{array}{rcrcrcrcrcc} & 1 & + & 2 & + & 3 & + & .... & + & 100 = & x \\ + & 100 & + & 99 & + & 98 & + & ... & + & 1 = & x \\ \hline & 101 & + & 101 & + & 101 & + & ... & + & 101 = & 2x \end{array}$$

The sum of the second column is $2+99 = 101$. The sum of the third column is $3+98 = 101$. And so on, the sum of each column is 101 because as we step to the right, the number in the first row increase by 1 and the number in the second row decrease by 1. Thus the sum remains 101.

$$\begin{array}{rrcrcrcccrcc} & 1 & + & 2 & + & 3 & + & .... & + & 100 & = & x \\ + & 100 & + & 99 & + & 98 & + & ... & + & 1 & = & x \\ \hline & 101 & + & 101 & + & 101 & + & ... & + & 101 & = & 2x \end{array}$$

When we add the same number to itself repeatedly, that can be re-written as multiplication. We have 100 columns, so we added 101 to itself 100 times. So the long sum can be replaced by a single multiplication.

$$\begin{aligned} 100 \cdot 101 &= 2x \\ 10\,100 &= 2x \\ 5050 &= x \end{aligned}$$

This method can be applied to arithmetic sequences..

Example 2. Find the sum $3+6+9+12+....+600$.

Solution: We will apply Gauss's method.

$$\begin{array}{ccccccccc} 3 & + & 6 & + & 9 & + & ... & + & 600 & = & x \\ 600 & + & 597 & + & 594 & + & ... & + & 3 & = & x \end{array}$$

The easy question is: what is the sum in each column? Clearly 603. It takes a bit more work to figure out how many columns are there. We label the numbers in the first row as 1, 2, 3, etc

$$\begin{array}{ccccccccccc} 1 & & 2 & & 3 & & & & ? & = & x \\ \downarrow & & \downarrow & & \downarrow & & & & \downarrow & & \\ 3 & + & 6 & + & 9 & + & ... & + & 600 & = & x \end{array}$$

The last number will have to get the label 200. So, we have 200 columns and each of them adds up to 603.

$$\begin{aligned} 603+603+603+...+603 &= 2x \\ 200 \cdot 603 &= 2x \\ 120\,600 &= 2x \\ 60\,300 &= x \end{aligned}$$

So the sum $3+6+9+...+600 = \boxed{60\,300}$.

Example 3. Suppose that an arithmetic sequence is defined by first term $a = 73$ and common difference $d = 15$. Find a_{120} and s_{120}.

Solution: We first find a_{120}

$$a_{120} = a + 119d = 73 + 119 \cdot 15 = 1858$$

For the sum s_n, we apply Gauss's method. We write down the sum twice, the second time backwards, and add, column by column.

$$\begin{array}{cccccccccccc} & 73 & + & 88 & + & 103 & + & .... & + & 1858 & = & x \\ + & 1858 & + & 1843 & + & 1828 & + & ... & + & 73 & = & x \\ \hline & 1931 & + & 1931 & + & 1931 & + & ... & + & 1931 & = & 2x \end{array}$$

We know that each column adds to 1931 and there are 120 columns. Therefore, the sum is

$$\begin{aligned} 2s_{120} &= 120 \cdot 1931 \\ s_{120} &= \frac{120 \cdot 1931}{2} = 115\,860 \end{aligned}$$

In general we can find s_n using this method. We write down the long sum twice, the second time backwards and add by columns.

$$\begin{array}{cccccccccccc} & a_1 & + & a_2 & + & a_3 & + & .... & + & a_n & = & x \\ + & a_n & + & a_{n-1} & + & a_{n-2} & + & ... & + & a_1 & = & x \\ \hline \end{array}$$

The sum in each column is the same: $a_1 + a_n$ and we have n columns. Therefore,

$$2s_n = n\,(a_1 + a_n)$$

and so $s_n = n \cdot \dfrac{a_1 + a_n}{2}$

number of columns

the sum in each column

$$s_n = \frac{n(a_1 + a_n)}{2}$$

fix the overcount

We can combine our two formulas, $a_n = a + (n-1)\,d$ and $s_n = \dfrac{n\,(a_1 + a_n)}{2}$. Because $a_1 = a$ and $a_n = a + (n-1)\,d$, we have that

$$s_n = \frac{n\,(a_1 + a_n)}{2} = \frac{n\,(a + a + (n-1)\,d)}{2} = n \cdot \frac{2a + (n-1)\,d}{2}$$

> If a is the first term and d the common difference of an arithmetic sequence, then a_n and $s_n = a_1 + a_2 + \ldots + a_n$ can be found using the formulas
>
> $$a_n = a + (n-1)\,d \quad \text{and} \quad s_n = n\frac{2a + (n-1)\,d}{2}$$

Sample Problems

1. Consider the arithmetic sequence (a_n) determined by $a_1 = 143$ and $d = -3$.

   a) Find the 220th element in the sequence. b) Find the sum of the first 220 elements.

2. Consider the arithmetic sequence $1,\ 4,\ 7,\ 10,\ 13, ...$

   a) Find the 200th element in the sequence. b) Find the sum of the first 200 elements.

3. Consider the arithmetic sequence determined by $a_1 = 45$ and $d = -5$.

   a) Find a_{150}. b) Find the sum $a_1 + a_2 + ... + a_{150}$.

4. Suppose that (a_n) is an arithmetic sequence. Find the values of a and d if we know that $a_{10} = 38$ and $a_{15} = 18$.

5. Suppose that (a_n) is an arithmetic sequence. Find the values of a and d if we know that $a_{15} = 62$ and $s_{20} = 700$.

6. The sum of the first five elements of an arithmetic sequence is -45. Find the value of the third element. (In short: find a_3 if $s_5 = -45$).

7. The sum of the first three elements in an arithmetic sequence is 219. The sum of the first nine elements in the same arithmetic sequence is 603. Find the 143rd element in this sequence.

8. Find the first element and common difference in an arithmetic sequence if we know that $s_{20} = 230$ and $s_{39} = -663$.

9. The first elements in an arithmetic sequence is 2, its twenty-second element is 14. Find the value of n so that $a_n = 6$.

10. The first eight elements in an arithmetic sequence add up to 604. The next eight elements add up to 156. Find the first element and common difference in the sequence.

11. The first element in an arithmetic sequence is 80. Find the common difference if we also know that s_9 is eighteen times a_{11}.

12. Given the arithmetic sequence by $a_1 = -16$ and $d = \frac{1}{3}$, find all values of n so that $s_n = 50$.

13. Suppose that $\{a_n\}$ is an arithmetic sequence with $a_1 = 1$. Find the second element if we know that the sum of the first five elements is a quarter of the sum of the next five elements.

14. Three sides of a right triangle are integers and form consecutive terms in an arithmetic sequence. Find the sides of the triangle.

15. The first element in an arithmetic sequence is 10. Find the common difference in the sequence such that a_5, a_{51}, and a_{55} are sides of a right triangle and a_{55} is the hypotenuse.

16. Consider the arithmetic sequence of odd natural numbers, $1, 3, 5, 7, 9, 11...$ Prove that for all natural numbers n, s_n is a perfect square.

17. Suppose that $a_1, a_2, a_3, ...$ and $b_1, b_2, b_3, ...$ are arithmetic sequences. The sequence $c_1, c_2, c_3, ...$ is formed by multiplying the two sequences term by term, i.e. $c_1 = a_1 b_1$, $c_2 = a_2 b_2$, ...Find the value of c_8 if we know that $c_1 = 10$, $c_2 = 48$, and $c_3 = 66$.

Practice Problems

1. Consider the arithmetic sequence determined by $a_1 = 8$ and $d = -3$. Find a_{20} and s_{20}.
2. Consider the arithmetic sequence $(a_n) = 78, 75, 72, 69, ...$Find a_{150} and s_{150}.
3. An arithmetic sequence is defined by $a_1 = 54$ and $d = -11$. Find a_{10} and s_{10}.
4. A theater has 30 rows of seats. The first row contains 20 sets, the secons row contains 21 seats, and so on, each row has one more seat than the previos one. How many seats are there in the theater?
5. Find a and d if $a_{34} = 193$ and $s_{17} = 306$.
6. The first seven elements of an arithmetic sequence add up to 91. The first fifteen elements add up to 495. Find the second element in the sequence.
7. Consider the arithmetic sequence (a_n) with the following conditions: $a_{50} = 252$ and $s_{50} = 2800$. Find the first element and common difference of the sequence.
8. Find a and d if

   a) $a_5 = 45$ and $a_{33} = 24$

   b) $a_{30} = 13$ and $s_{30} = -480$

   c) $a_{25} = 25$ and $s_{45} = 1170$

   d) $s_{36} = 288$ and $s_{99} = -5445$
9. The fifth elements in an arithmetic sequence is -1, and its twenty-first element is 11. Find the value of n so that $a_n = 20$.
10. The first element in an arithmetic sequence is 4. Find the common difference in the sequence if given that a_{10}, a_{31}, and a_{34} are sides of a right triangle where a_{34} is the hypotenuse.
11. Two arithmetic sequences are multiplied together to produce the sequence 468, 462, 384, ... What is the next term of this sequence?

Answers

Sample Problems

1. a) -514 b) $-40\,810$ 2. a) 598 b) $59\,900$ 3. a) -700 b) $-49\,125$ 4. $a = 74$, $d = -4$
5. $a = -22$, $d = 6$ 6. -9 7. -209 8. $a = 40$ and $d = -3$ 9. the 8th element 10. $a = 100$, $d = -7$ 11. -5
12. 100 13. -2 14. $3d$, $4d$, and $5d$, where d is any positive number 15. $\dfrac{1}{2}$ 16. see solutions 17. -144

Practice Problems

1. $a_{20} = -49$ and $s_{20} = -410$ 2. $a_{150} = -369$ and $s_{150} = -21\,825$ 3. $a_{10} = -45$, $d = 45$ 4. 1035 seats
5. $a = -38$, $d = 7$ 6. 3 7. $a = -140$, $d = 8$
8. a) $a = 48, d = -\dfrac{3}{4}$ b) $a = -45, d = 2$ c) $a = 37, d = -\dfrac{1}{2}$ d) $a = 43, d = -2$ 9. $n = 33$ 10. $\dfrac{2}{3}$ 11. 234

Sample Problems - Solutions

Please note that the first element is denoted by both a and a_1.

1. Consider the arithmetic sequence (a_n) determined by $a_1 = 143$ and $d = -3$.

   a) Find the 220th element in the sequence.

   Solution: $a_{220} = a_1 + 219d = 143 + 219(-3) = -514$

   b) Find the sum of the first 220 elements.

   Solution 1 : We can use the formula $s_n = \dfrac{a_1 + a_n}{2}(n)$. We set $n = 220$.

$$s_{220} = \frac{a_1 + a_{220}}{2}(220) = \frac{143 + (-514)}{2}(220) = \frac{-371}{2} \cdot 220 = -40\,810$$

   Solution 2: We can use the formula $s_n = \dfrac{2a + (n-1)d}{2}(n)$. We set $n = 220$.

$$s_{220} = \frac{2(143) + 219(-3)}{2}(220) = \frac{-371}{2} \cdot 220 = -40\,810$$

2. Consider the arithmetic sequence $1,\ 4,\ 7,\ 10,\ 13, \ldots$

   a) Find the 200th element in the sequence.

   Solution: We see that $d = 3$. Then $a_{200} = a_1 + 199d = 1 + 199(3) = 598$

   b) Find the sum of the first 200 elements.

   Solution 1 : $s_{200} = \dfrac{a_1 + a_{200}}{2}(200) = \dfrac{1 + 598}{2}(200) = 59\,900$

   Solution 2: $s_{200} = \dfrac{2a + 199d}{2}(200) = \dfrac{2(1) + 199(3)}{2}(200) = 59\,900$

3. Consider the arithmetic sequence determined by $a_1 = 45$ and $d = -5$.

   a) Find a_{150}.

   Solution: $a_{150} = a_1 + 149d = 45 + 149(-5) = -700$

   b) Find the sum $a_1 + a_2 + \ldots + a_{150}$.

   Solution 1 : $s_{150} = \dfrac{a_1 + a_{150}}{2}(150) = \dfrac{45 + (-700)}{2}(150) = -49\,125$

   Solution 2: $s_{150} = \dfrac{2a + 149d}{2}(150) = \dfrac{2(45) + 149(-5)}{2}(150) = -49\,125$

4. Suppose that (a_n) is an arithmetic sequence. Find the values of a and d if we know that $a_{10} = 38$ and $a_{15} = 18$.

   Solution 1: Notice that the 15th element is smaller than the tenth: this sequence is decreasing, and so d must be negative. Let a denote the first element and d the common difference in the arithmetic sequence. $a_{10} = 38$ and $a_{15} = 18$ can be expressed as

$$\begin{cases} a + 9d = 38 \\ a + 14d = 18 \end{cases}$$

   We will sove this system by elimination; we multiply the first equation by -1.

$$\begin{aligned} -a - 9d &= -38 \\ a + 14d &= 18 \end{aligned}$$

   Then add the two equations.

$$\begin{aligned} 5d &= -20 \\ d &= -4 \end{aligned}$$

$$\begin{aligned} a+9(-4) &= 38 \\ a-36 &= 38 \\ a &= 74 \end{aligned}$$

Thus $a=74$ and $d=-4$. We check: $a_{10}=a+9d=74+9(-4)=38$ and $a_{15}=a+14d=74+14(-4)=18$.
Solution 2. A neat shortcut:

$$\begin{aligned} a_{15} &= a_{10}+5d \\ 18 &= 38+5d \\ -20 &= 5d \\ -4 &= d \end{aligned}$$

The rest of the solution is identical to the previous method.

5. Suppose that (a_n) is an arithmetic sequence. Find the values of a and d if we know that $a_{15}=62$ and $s_{20}=700$.
Solution: Let a denote the first element and d the common difference in the arithmetic sequence. $a_{15}=a+14d$, and $a_{20}=a+19d$. Then

$$a_{15} = a+14d \qquad \Longrightarrow \qquad 62=a+14d$$

$$s_{20} = \frac{2a+19d}{2}(20) \qquad \Longrightarrow \qquad 700=(2a+19d)\,10$$

The second equation can be further simplified by division by 10. Now we have the system

$$\begin{cases} a+14d=62 \\ 2a+19d=70 \end{cases}$$

We solve this system by elimination; we multiply the first equation by -2 and then add the two equations.

$$\begin{aligned} -2a-28d &= -124 \\ 2a+19d &= 70 \\ -9d &= -54 \\ d &= 6 \end{aligned}$$

$$\begin{aligned} a+14(6) &= 62 \\ a+84 &= 62 \\ a &= -22 \end{aligned}$$

Thus $a=-22$ and $d=6$. We check: $a_{15}=-22+14(6)=62$, $a_{20}=-22+19(6)=92$, and so $s_{20}=\dfrac{a_1+a_{20}}{2}(20)=\dfrac{-22+92}{2}(20)=700$.

6. The sum of the first five elements of an arithmetic sequence is -45. Find the value of the third element. (In short: find a_3 if $s_5=-45$)
Solution 1: Using the usual notation, a and d, we need to find the value of $a_3=a+2d$, and we have that

$$\begin{aligned} a_1+a_2+a_3+a_4+a_5 &= -45 \\ a+(a+d)+(a+2d)+(a+3d)+(a+4d) &= -45 \\ 5a+10d &= -45 \\ 5(a+2d) &= -45 \\ a+2d &= -9 \\ a_3 &= -9 \end{aligned}$$

Solution 2: Let x denote the third term. Then the first five elements are

$$x-2d, \quad x-d, \quad x, \quad x+d, \quad x+2d$$

and if we add these, we easily get $5x$. Thus $5x=-45$ gives us $x=-9$. If we select any odd number of consecutive elements in an arithmetic sequence, the middle element will be the average (arithmetic mean) of these elements. It is often useful to use this in notation: three consecutive elements in an arithmetic sequence can be denoted as $x-d$, x, and $x+d$.

7. The sum of the first three elements in an arithmetic sequence is 219. The sum of the first nine elements in the same arithmetic sequence is 603. Find the 143rd element in this sequence.
Solution1: We simply state the two partial sums and solve the system of linear equation for a and d. However, the equations of this system can be significantly simplified, for a very good reason. For more on this, see the second solution presented.

$$\begin{aligned} s_3 &= 219 \quad \Longrightarrow \quad 219=\frac{2a+2d}{2}(3) \quad \Longrightarrow \quad a+d=73 \\ s_9 &= 603 \quad \Longrightarrow \quad 603=\frac{2a+8d}{2}(9) \quad \Longrightarrow \quad a+4d=67 \end{aligned}$$

We solve the system

$$\begin{cases} a+d=73 \\ a+4d=67 \end{cases}$$

and obtain $a=75$ and $d=-2$. Then the 143rd element can be easily found

$$a_{143}=a+142d=75+142(-2)=-209$$

Solution 2. The average (or arithmetic mean) of an odd number of consecutive elements in an arithmetic sequence is always the element in the middle. If we add three consecutive elements in an arithmetic sequence, the sum is always three times the middle element. If we add nine consecutive elements in an arithmetic sequence, the sum is always nine times the middle element. Using this fact, we almost immediately have the following:

$$\begin{aligned} s_3 &= 219 \quad \Longrightarrow \quad 219=3a_2 \quad \Longrightarrow \quad a_2=73 \\ s_9 &= 603 \quad \Longrightarrow \quad 603=9a_5 \quad \Longrightarrow \quad a_5=67 \end{aligned}$$

Notice that the statements $a_2=73$ and $a_5=67$ are the same as the equations we obtained in Solution 1. We can now easily solve for a and d.

$$\begin{aligned} a_5 &= a_2+3d \\ 67 &= 73+3d \quad \Longrightarrow \quad d=-2 \end{aligned}$$

and $a_2=73$ gives us

$$\begin{aligned} a_2 &= a+d \\ 73 &= a-2 \quad \Longrightarrow \quad a=75 \end{aligned}$$

8. Find the first element and common difference in an arithmetic sequence if we know that $s_{20}=230$ and $s_{39}=-663$.
Solution: Let a and d denote the first element and common difference in the sequence. Then we will set up two equations in a and d stating $s_{20}=230$ and $s_{39}=-663$. Recall the formula $s_n=\dfrac{2a+(n-1)\,d}{2}\cdot n$.

$$\begin{aligned} s_{20} &= 230 \\ \frac{2a+19d}{2}\cdot 20 &= 230 \qquad \text{simplify} \\ (2a+19d)\cdot 10 &= 230 \qquad \text{divide by 10} \\ 2a+19d &= 23 \end{aligned}$$

$$\begin{aligned} s_{39} &= -663 \\ \frac{2a+38d}{2}\cdot 39 &= -663 \qquad \text{divide by 39} \\ \frac{2a+38d}{2} &= -17 \qquad \text{multiply by 2} \\ 2a+38d &= -34 \end{aligned}$$

So now we just need to solve the system of equations $\begin{cases} 2a+19d=23 \\ 2a+38d=-34 \end{cases}$. We will leave this task for the reader. The system's solution is $a=40$ and $d=-3$.

9. The first elements in an arithmetic sequence is 2, its twenty-second element is 14. Find the value of n so that $a_n=6$.

Solution: Let a and d denote the first element and common difference of the sequence. We will first solve for d.

$$\begin{aligned} a_{22} &= 14 \\ a+21d &= 14 \qquad \text{we know } a=2 \\ 2+21d &= 14 \\ 21d &= 12 \\ d &= \frac{12}{21}=\frac{4}{7} \end{aligned}$$

We will now solve for n in $a_n=6$

$$\begin{aligned} a_n &= 6 \\ a+(n-1)\,d &= 6 \qquad \text{we know } a=2 \text{ and } d=\frac{4}{7} \\ 2+(n-1)\,\frac{4}{7} &= 6 \qquad \text{subtract } 2 \\ (n-1)\,\frac{4}{7} &= 4 \qquad \text{divide by } 4 \\ (n-1)\,\frac{1}{7} &= 1 \qquad \text{multiply by } 7 \\ n-1 &= 7 \qquad \text{add } 1 \\ n &= 8 \end{aligned}$$

So the eighth element is 6.

10. The first eight elements in an arithmetic sequence add up to 604. The next eight elements add up to 156. Find the first element and common difference in the sequence.

Solution: At first this problem looks tricky because the sum of the second eight elements seem to be a useless piece of information. But it quickly becomes routine once we realize that the sum of the forst eight and second eight elements in the sequence, if we add them, simply gives us the sum of the first sixteen elements. In short, $s_{16}=604+156=760$. Thus we have the system

$$\begin{array}{lll} s_8 = 604 & \dfrac{2a+7d}{2}(8)=604 & \Longrightarrow \quad 2a+7d=151 \\ s_{16} = 760 & \dfrac{2a+15d}{2}(16)=760 & \Longrightarrow \quad 2a+15d=95 \end{array}$$

We solve the system of linear equations for a and d. We will multiply the first equation by -1 and then add the two equations.

$$\begin{aligned} -2a-7d &= -151 \\ 2a+15d &= 95 \end{aligned}$$

$$\begin{aligned} 8d &= -56 \\ d &= -7 \end{aligned}$$

Now we substitute $d=-7$ into either one of the two equations and solve for a. We easily obtain $a=100$.

11. The first element in an arithmetic sequence is 80. Find the common difference if we also know that s_9 is eighteen times a_{11}.

Solution: Recall that $a_n = a + (n-1)\,d$ and $s_n = \dfrac{2a + (n-1)\,d}{2}n$.

$$\begin{aligned}
s_9 &= 18a_{11} \\
\frac{2\,(80) + 8d}{2}(9) &= 18\,(80 + 10d) && \text{divide by 9} \\
\frac{2\,(80) + 8d}{2} &= 2\,(80 + 10d) \\
\frac{2\,[(80) + 4d]}{2} &= 2\,(80 + 10d) && \text{simplify} \\
80 + 4d &= 2\,(80 + 10d) && \text{distribute} \\
80 + 4d &= 160 + 20d && \text{subtract } 4d \\
80 &= 160 + 16d && \text{subtract 160} \\
-80 &= 16d && \text{divide by 16} \\
-5 &= d
\end{aligned}$$

12. Given the arithmetic sequence by $a_1 = -16$ and $d = \dfrac{1}{3}$, find all values of n so that $s_n = 50$.

Solution: Recall that $s_n = \dfrac{2a + (n-1)\,d}{2}n$ We write $a = -16$, $d = \dfrac{1}{3}$, and $s_n = 50$, and solve for n.

$$\begin{aligned}
s_n &= \frac{2a + (n-1)\,d}{2}n \\
50 &= \frac{2\,(-16) + (n-1)\,\frac{1}{3}}{2}n && \text{multiply by 2} \\
100 &= \left(-32 + (n-1)\,\frac{1}{3}\right)n && \text{distribute } n \\
100 &= -32n + n\,(n-1)\,\frac{1}{3} && \text{multiply by 3} \\
300 &= -96n + n\,(n-1) \\
300 &= -96n + n^2 - n \\
0 &= n^2 - 97n - 300
\end{aligned}$$

We will solve this quadratic equation using the quadratic formula.

$$n_{1,2} = \frac{97 \pm \sqrt{97^2 - 4\,(-300)}}{2} = \frac{97 \pm \sqrt{9409 + 1200}}{2} = \frac{97 \pm \sqrt{10\,609}}{2} = \frac{97 \pm 103}{2} = \begin{cases} \dfrac{200}{2} = 100 \\ \dfrac{-6}{2} = -3 \end{cases}$$

Since n represents the index of the element in a sequence, it can not be a negative number, and so -3 is ruled out. The answer is 100.

13. Suppose that (a_n) is an arithmetic sequence with $a_1 = 1$. Find the second element if we know that the sum of the first five elements is a quarter of the sum of the next five elements.

Solution: Let d denote the common difference. If S denotes the sum of the first five elements, i.e. $S = s_5$, then the second five elements add up to $4S$. Thus $s_{10} = S + 4S = 5S$. Thus we have that the sum of the first 10 elements is five times the sum of the first ten elements.

$$\begin{aligned}
5s_5 &= s_{10} \\
5\left(\frac{2 + 4d}{2}\right)(5) &= \frac{2 + 9d}{2}(10) && \text{multiply by 2}
\end{aligned}$$

$$\begin{aligned} 25(4d+2) &= 10(9d+2) \\ 100d+50 &= 90d+20 && \text{subtract } 90d \\ 10d+50 &= 20 && \text{subtract } 50 \\ 10d &= -30 && \text{divide by } 10 \\ d &= -3 \end{aligned}$$

Thus the second element is $a+d=1-3=-2$.

14. Three sides of a right triangle are integers and form consecutive terms in an arithmetic sequence. Find the sides of the triangle.

    Solution: Let us denote the middle side by x. Then the shortest side is $x-d$ and the longest is $x+d$. The Pythagorean Theorem states then that

$$\begin{aligned} (x-d)^2+x^2 &= (x+d)^2 \\ x^2-2xd+d^2+x^2 &= x^2+2dx+d^2 \\ x^2-4xd &= 0 \\ x(x-4d) &= 0 \end{aligned}$$

    Either $x=0$ (impossible for a side of a triangle) or $x=4d$. Then the three sides are $3d$, $4d$, and $5d$. Which means that all such triangles are similar to the triangle with sides 3, 4, and 5 units.

15. The first element in an arithmetic sequence is 10. Find the common difference in the sequence such that a_5, a_{51}, and a_{55} are sides of a right triangle and a_{55} is the hypotenuse.

    Solution: We express a_5, a_{51}, and a_{55} in terms of a and d. It may be useful to note it now that if these elements are sides of a rigth triangle in this order, then d must be positive.

$$\begin{aligned} a_5 &= a+4d=10+4d \\ a_{51} &= a+50d=10+50d \\ a_{55} &= a+54d=10+54d \end{aligned}$$

    We write the Pythagorean Theorem for these three quantities

$$\begin{aligned} (a_5)^2+(a_{51})^2 &= (a_{55})^2 \\ (10+4d)^2+(10+50d)^2 &= (10+54d)^2 \end{aligned}$$

    We solve this quadratic equation for d.

$$\begin{aligned} (10+4d)^2+(10+50d)^2 &= (10+54d)^2 \\ 100+80d+16d^2+100+1000d+2500d^2 &= 100+1080d+2916d^2 \\ 2516d^2+1080d+200 &= 2916d^2+1080d+100 \\ 0 &= 400d^2-100 \\ 0 &= 100\left(4d^2-1\right) \\ 0 &= 100(2d+1)(2d-1) \\ d &= \pm\frac{1}{2} \end{aligned}$$

    Since d must be positive, $d=\dfrac{1}{2}$. We check: If $a=10$ and $d=\dfrac{1}{2}$, then

$$\begin{aligned} a_5 &= a+4d=10+4\left(\frac{1}{2}\right)=12 \text{ and } a_{51}=a+50d=10+50\left(\frac{1}{2}\right)=35 \quad \text{and} \\ a_{55} &= a+54d=10+54\left(\frac{1}{2}\right)=37 \end{aligned}$$

    and $12^2+35^2=37^2$ and so our solution is correct.

16. Consider the arithmetic sequence of odd natural numbers, $1, 3, 5, 7, 9, 11...$ Prove that for all n, s_n is a perfect square.

Proof: Clearly $a = 1$ and $d = 2$.

$$s_n = \frac{2a + (n-1)\,d}{2}(n) = \frac{2 \cdot 1 + (n-1)\,2}{2}(n) = \frac{2 + 2n - 2}{2}(n) = \frac{2n}{2}(n) = n^2$$

Not only these sums are all squares, but actually $s_n = n^2$.

17. Suppose that $a_1, a_2, a_3, ...$ and $b_1, b_2, b_3, ...$ are arithmetic sequences. The sequence $c_1, c_2, c_3, ...$ is formed by multiplying the two sequences term by term, i.e. $c_1 = a_1 b_1$, $c_2 = a_2 b_2$, ...Find the value of c_8 if we know that $c_1 = 10$, $c_2 = 48$, and $c_3 = 66$.

Solution: Let a and d_1 denote the first element and difference in the arithmetic sequence (a_n) and b and d_2 in (b_n). We are given that

$$\begin{aligned} c_1 &= a_1 b_1 = 10 \\ c_2 &= a_2 b_2 = 48 \\ c_3 &= a_3 b_3 = 66 \end{aligned}$$

In terms of a, b, d_1 and d_2:

$$\begin{aligned} ab &= 10 \\ (a + d_1)(b + d_2) &= 48 \\ (a + 2d_1)(b + 2d_2) &= 66 \end{aligned}$$

Since we have four unknowns and only three equations, it seems that we can not find the value of all unknowns. In this case, we also need to keep an eye on what we need to find:

$$c_8 = a_8 b_8 = (a + 7d_1)(b + 7d_2) = ab + 7ad_2 + 7bd_1 + 49d_1 d_2$$

We perform the multiplications in each equation

$$\begin{aligned} ab &= 10 \\ ab + ad_2 + bd_1 + d_1 d_2 &= 48 \\ ab + 2ad_2 + 2bd_1 + 4d_1 d_2 &= 66 \\ ab + 7ad_2 + 7bd_1 + 49d_1 d_2 &= ? \end{aligned}$$

We substitute $ab = 10$ into eaqch equation and simplify

$$\begin{aligned} (ad_2 + bd_1) + (d_1 d_2) &= 38 \\ 2(ad_2 + bd_1) + 4(d_1 d_2) &= 56 \\ 10 + 7(ad_2 + bd_1) + 49(d_1 d_2) &= ? \end{aligned}$$

If we introduce the new variables $X = ad_2 + bd_1$ and $Y = d_1 d_2$, we can solve the system of linear equations in two variables:

$$\begin{aligned} X + Y &= 38 \\ 2X + 4Y &= 56 \quad \text{and} \quad 10 + 7X + 49Y = ? \end{aligned}$$

We easily solve this and obtain $X = 48$ and $Y = -10$. Now we can compute c_8.

$$c_8 = 10 + 7X + 49Y = 10 + 7(48) + 49(-10) = -144$$

For more documents like this, visit our page at https://teaching.martahidegkuti.com and click on Lecture Notes. E-mail questions or comments to mhidegkuti@ccc.edu.