Sample Problems

1. Find the coordinates of all points where the circle $x^2 + (y-3)^2 = 50$ and the line $y = -2x + 8$ intersect each other.

2. a) Find the points where the circles $(x+4)^2 + (y+1)^2 = 10$ and $(x+1)^2 + (y-5)^2 = 25$ intersect each other.
   b) Find the points where the circles $(x+2)^2 + (y+2)^2 = 50$ and $(x-2)^2 + (y-1)^2 = 25$ intersect each other.

3. Consider the circles $x^2 + y^2 = 36$ and $(x-5)^2 + y^2 = 16$. Let C be the intersection of the two common tangent lines drawn to the circle.
   a) Find the coordinates of C.
   b) Consider one of the common tangent lines. Find the distance between the points of tangency.

4. Find an equation of the circle that passes through the points $A(0,2)$, $B(-2,-2)$, and $C(-8,-4)$.

Practice Problems

1. Find the coordinates of all point(s) where the circle and the line intersect each other.
   a) $(x+5)^2 + (y-7)^2 = 8$ and $x + y = 6$
   b) $(x-2)^2 + (y+1)^2 = 50$ and $y = -x + 9$
   c) $x^2 + (y+4)^2 = 25$ and $y = x - 24$
   d) $(x-3)^2 + (y+1)^2 = 25$ and $y = -x + 1$

2. Find the coordinates of all points where the given circles intersect each other.
   a) $(x-1)^2 + (y-2)^2 = 10$ and $(x-10)^2 + (y-5)^2 = 40$
   b) $(x-3)^2 + y^2 = 20$ and $x^2 + (y-1)^2 = 50$
   c) $x^2 + (y-8)^2 = 26$ and $(x-7)^2 + (y-3)^2 = 4$

3. Consider the circles $(x+5)^2 + y^2 = 100$ and $x^2 + y^2 = 49$. Let C be the intersection of the two common tangent lines drawn to the circle.
   a) Find the coordinates of C.
   b) Consider one of the common tangent lines. Find the distance between the points of tangency.

4. Let C_1 and C_2 be circles defined by $x^2 + (y+7)^2 = 1$ and $x^2 + (y-6)^2 = 16$, respectively. Consider one of the common tangent lines. Find the distance between the points of tangency.

5. Given the three points, find an equation of the circle that contains all three points.
   a) $P(0,0)$, $Q(0,8)$, and $R(-6,0)$.
   b) $A(-1,6)$, $B(-7,12)$, and $C(-3,0)$.
   c) $X(3,-7)$, $Y(5,-1)$, and $Z(-3,-5)$.

Sample Problems - Answers

1. $(-1,10)$ and $(5,-2)$ 2. a) $(-1,0)$ and $(-5,2)$ b) $(-1,5)$ and $(5,-3)$

3. a) $(15,0)$ b) $\sqrt{21}$ 4. $(x+7)^2+(y-3)^2=50$

Practice Problems - Answers

1. a) $(-3,9)$ b) $(3,6)$ and $(9,0)$ c) they don't intersect d) $(-1,2)$ and $(6,-5)$

2. a) $(4,3)$ b) $(7,2)$ and $(5,-4)$ c) the circles do not intersect

3. a) $\left(\dfrac{35}{3},0\right)$ b) 4 4. $4\sqrt{10}$ or 12 5. a) $(x+3)^2+(y-4)^2=25$

b) $(x+8)^2+(y-5)^2=50$ c) $(x-1)^2+(y+3)^2=20$

Sample Problems - Solutions

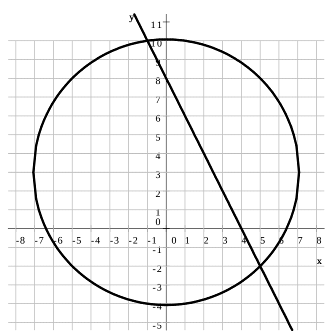
1. Find the coordinates of all points where the circle $x^2 + (y-3)^2 = 50$ and the line $y = -2x + 8$ intersect each other.
Solution: We need to solve the following system of equations:

$$\begin{cases} x^2 + (y-3)^2 = 50 \\ y = -2x + 8 \end{cases}$$

We will use substitution. We substitute $y = -2x + 8$ in the first equation and solve for x.

$$\begin{aligned} x^2 + (-2x+8-3)^2 &= 50 \\ x^2 + (-2x+5)^2 &= 50 \\ x^2 + 4x^2 - 20x + 25 &= 50 \\ 5x^2 - 20x + 25 &= 50 \end{aligned} \qquad \begin{aligned} &5x^2 - 20x - 25 = 0 \\ &5\left(x^2 - 4x - 5\right) = 0 \\ &5(x-5)(x+1) = 0 \\ &x_1 = 5 \qquad x_2 = -1 \end{aligned}$$

We now use the second equation to find the $y-$value belonging to the $x-$values we just obtained. If $x = 5$, then $y = -2(5) + 8 = -2$. If $x = -1$, then $y = -2(-1) + 8 = 10$. Thus the two points are $(5, -2)$ and $(-1, 10)$. We check: both points should be on both the circle and the line.



2. a) Find the points where the circles $(x+4)^2 + (y+1)^2 = 10$ and $(x+1)^2 + (y-5)^2 = 25$ intersect each other.
Solution: We need to solve the following system:

$$\begin{cases} (x+4)^2 + (y+1)^2 = 10 \\ (x+1)^2 + (y-5)^2 = 25 \end{cases}$$

We multiply out the complete squares and combine like terms in both equations.

$$\begin{aligned} x^2 + 8x + 16 + y^2 + 2y + 1 &= 10 \\ x^2 + 8x + y^2 + 2y + 17 &= 10 \\ x^2 + 8x + y^2 + 2y &= -7 \end{aligned} \qquad \begin{aligned} x^2 + 2x + 1 + y^2 - 10y + 25 = 25 \\ x^2 + 2x + y^2 - 10y + 26 = 25 \\ x^2 + 2x + y^2 - 10y = -1 \end{aligned}$$

We will multiply the second equation by -1 and add the two equations. This will cancel out all quadratic terms.

$$\begin{aligned} x^2 + 8x + y^2 + 2y &= -7 \\ -x^2 - 2x - y^2 + 10y &= 1 \end{aligned}$$

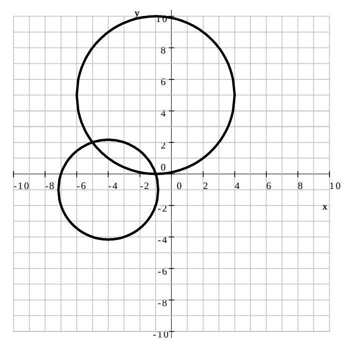
The sum of the two equations is

$$\begin{aligned} 6x + 12y &= -6 \qquad \text{divide both sides by 6} \\ x + 2y &= -1 \\ x &= -2y - 1 \end{aligned}$$

We substitute this into the first equation and solve for y.

$$\begin{aligned}
(x+4)^2+(y+1)^2 &= 10 \quad \text{and} \quad x=-2y-1 \\
(x+4)^2+(y+1)^2 &= 10 \\
\Big(\underbrace{-2y-1}_{x}+4\Big)^2+(y+1)^2 &= 10 \\
(-2y+3)^2+(y+1)^2 &= 10 \\
4y^2-12y+9+y^2+2y+1 &= 10 \\
5y^2-10y+10 &= 10 \\
5y^2-10y &= 0 \\
5y\,(y-2) &= 0 \qquad \Longrightarrow \quad y_1=0 \qquad y_2=2
\end{aligned}$$

We now find the x values belonging to the $y-$values, using $x=-2y-1$. If $y_1=0$, then $x_1=-2\cdot 0-1=-1$ and if $y_2=2$, then $x_2=-2\cdot 2-1=-5$. Thus the two circles intersect at the points $(-1,0)$ and $(-5,2)$.



b) Find the points where the circles $(x+2)^2+(y+2)^2=50$ and $(x-2)^2+(y-1)^2=25$ intersect each other.
Solution: We need to solve the following system:

$$\begin{cases} (x+2)^2+(y+2)^2=50 \\ (x-2)^2+(y-1)^2=25 \end{cases}$$

We multiply out the complete squares and combine like terms in both equations

$$\begin{aligned}
x^2+4x+4+y^2+4y+4 &= 50 \qquad\qquad x^2-4x+4+y^2-2y+1=25 \\
x^2+4x+y^2+4y &= 42 \qquad\qquad x^2-4x+y^2-2y=20
\end{aligned}$$

We will multiply the second equation by -1 and add the two equations. This will cancel out all quadratic terms.

$$\begin{aligned}
x^2+4x+y^2+4y &= 42 \\
-x^2+4x-y^2+2y &= -20
\end{aligned}$$

The sum of the two equations is

$$\begin{aligned}
8x+6y &= 22 \quad \text{divide both sides by 2} \\
4x+3y &= 11
\end{aligned}$$

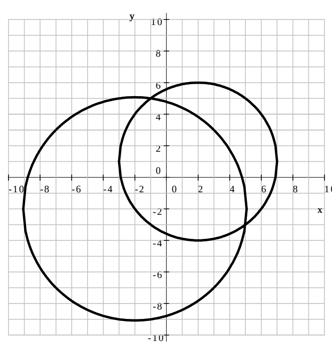
We now solve for y

$$y=\frac{-4x+11}{3}=-\frac{4}{3}x+\frac{11}{3}$$

We substitute this into the first equation and solve for x.

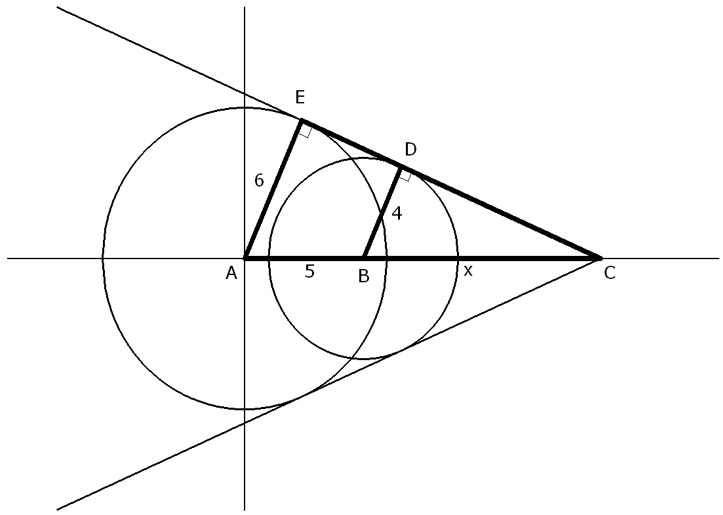
$$\begin{aligned}
(x+2)^2+(y+2)^2 &= 50 \qquad \text{and} \qquad y=-\frac{4}{3}x+\frac{11}{3}\\
(x+2)^2+\Bigg(\underbrace{-\frac{4}{3}x+\frac{11}{3}}_{y}+2\Bigg)^2 &= 50 \qquad \frac{11}{3}+2=\frac{17}{3}\\
(x+2)^2+\left(\frac{-4x+17}{3}\right)^2 &= 50\\
(x+2)^2+\frac{(-4x+17)^2}{3^2} &= 50 \qquad \text{since } \left(\frac{a}{b}\right)^2=\frac{a^2}{b^2}\\
(x+2)^2+\frac{(-4x+17)^2}{9} &= 50 \qquad \text{multiply by 9}\\
9(x+2)^2+(-4x+17)^2 &= 450\\
9\left(x^2+4x+4\right)+16x^2-136x+289 &= 450\\
9x^2+36x+36+16x^2-136x+289 &= 450\\
25x^2-100x-125 &= 0\\
25\left(x^2-4x-5\right) &= 0\\
25(x-5)(x+1) &= 0 \qquad \Longrightarrow \quad x_1=5 \qquad x_2=-1
\end{aligned}$$

We now find the y values belonging to the $x-$values, using $y=\dfrac{-4x+11}{3}$. If $x=5$, then $y=\dfrac{-4(5)+11}{3}=-3$ and if $x=-1$, then $y=\dfrac{-4(-1)+11}{3}=5$. Thus the two circles intersect at the points $(-1,5)$ and $(5,-3)$.

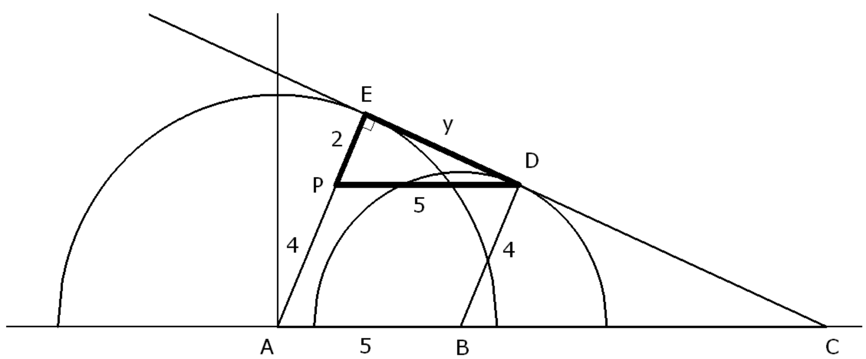


3. Consider the circles $x^2 + y^2 = 36$ and $(x-5)^2 + y^2 = 16$. Let C be the intersection of the two common tangent lines drawn to the circle.

   a) Find the coordinates of C.
   Solution: Using the notation shown on the picture below, triangles ACE and BCD are similar. Based on that, we write the equation $\frac{6}{4} = \frac{x+5}{x}$. The solution of this equation is 10. Thus C is $(15, 0)$.



   b) Consider one of the common tangent lines. Find the distance between the points of tangency.
   Solution: We draw a horizontal line through the point D. Let us denote the intersection of AE and the horizontal line by P. The quadrilateral $ABDP$ is a parallelogram since opposite sides are parallel. Consequently, opposite sides are equally long. This means that $AP = 4$ and so $PE = 2$. Also, $PD = 5$. We can now compute y via the Pythagorean theorem: $y = \sqrt{21}$.



4. Find an equation of the circle that passes through the points $A(0,2)$, $B(-2,-2)$, and $C(-8,-4)$.

We will present two different solutions.
Solution 1. This solution is based on the following fact. The **perpendicular bisector of a line segment** AB is the set of all points in the plane that are equidistant to points A and B. Recall that the center of the circle is equidistant to all points on the circle. Recall that the line segment connecting two points on a circle is call a chord. The center of a circle is contained on the perpendicular bisector of any of its chords.
Step 1. Let us find the equation of the perpendicular bisector of line segment AB. It is a line that is perpendicular to AB and passes through its midpoint, M.

$$m_{AB} = \frac{y_A - y_B}{x_A - x_B} = \frac{2-(-2)}{0-(-2)} = \frac{4}{2} = 2 \quad \Longrightarrow \quad m_{\text{perpendicular}} = -\frac{1}{2}$$

and the midpoint M can be found as

$$x_M = \frac{x_A + x_B}{2} = \frac{0+(-2)}{2} = -1 \quad \text{and} \quad y_M = \frac{y_A + y_B}{2} = \frac{2+(-2)}{2} = 0 \qquad M_{AB}(-1,0)$$

Thus the perpendicular bisector of AB is a line with slope $-\frac{1}{2}$, passing through $(-1,0)$. We can easily find this equation:

$$y = -\frac{1}{2}(x+1) = -\frac{1}{2}x - \frac{1}{2}$$

Step 2. We repeat the entire procedure with line segment AC.

$$m_{AC} = \frac{y_A - y_C}{x_A - x_C} = \frac{2-(-4)}{0-(-8)} = \frac{6}{8} = \frac{3}{4} \quad \Longrightarrow \quad m_{\text{perpendicular}} = -\frac{4}{3}$$

and the midpoint M can be found as

$$x_M = \frac{x_A + x_C}{2} = \frac{0+(-8)}{2} = -4 \quad \text{and} \quad y_M = \frac{y_A + y_C}{2} = \frac{2+(-4)}{2} = -1 \qquad M_{AC}(-4,-1)$$

Thus the perpendicular bisector of AC is a line with slope $-\frac{4}{3}$, passing through $(-4,-1)$. We can easily find this equation:

$$\begin{aligned} y+1 &= -\frac{4}{3}(x+4) \\ y &= -\frac{4}{3}(x+4) - 1 = -\frac{4}{3}x - \frac{16}{3} - 1 = -\frac{4}{3}x - \frac{19}{3} \end{aligned}$$

Step 3. The center of the circle lies on both perpendicular bisectors found in Steps 1 and 2. Thus it must be the intersection of those two lines. We find the intersection point by solving the system

$$\begin{aligned} y &= -\frac{1}{2}x - \frac{1}{2} \\ y &= -\frac{4}{3}x - \frac{19}{3} \end{aligned}$$

We can easily use substitution and obtain the equation

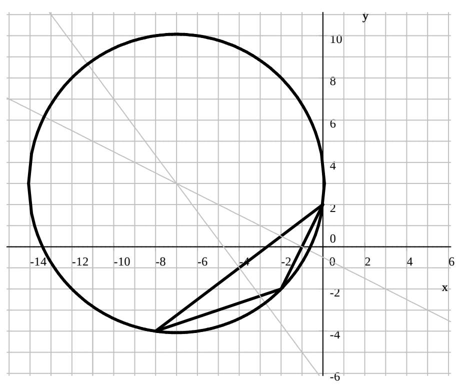
$$\begin{aligned} -\frac{1}{2}x - \frac{1}{2} &= -\frac{4}{3}x - \frac{19}{3} \qquad \text{multiply by 6} \\ -3x - 3 &= -8x - 38 \\ 5x - 3 &= -38 \\ 5x &= -35 \\ x &= -7 \qquad y = -\frac{1}{2}x - \frac{1}{2} = -\frac{1}{2}(-7) - \frac{1}{2} = \frac{7}{2} - \frac{1}{2} = 3 \end{aligned}$$

and obtain $(-7, 3)$.

Step 4. To find the radius, we need to compute the distance between $(-7, 3)$ and any of the points A, B, and C. Let us compute the distance of $(-7, 3)$ from A.

$$r = \sqrt{(-7 - x_A)^2 + (3 - y_A)^2} = \sqrt{(-7 - 0)^2 + (3 - 2)^2} = \sqrt{50}$$

Thus the equation of the circle is $(x + 7)^2 + (y - 3)^2 = 50$.



Solution 2. Suppose that the cicle has an equation

$$(x-k)^2+(y-h)^2=r^2$$

To find this equation, we must find the real numbers k, h, and r^2. Since there are three unknown variables, we will need three equations. We obtain those by stating that points A, B, and C are all on the circle.

$$\begin{array}{rrcll} 1) & (0-k)^2+(2-h)^2 & = & r^2 & A(0,2) \\ 2) & (-2-k)^2+(-2-h)^2 & = & r^2 & B(-2,-2) \\ 3) & (-8-k)^2+(-4-h)^2 & = & r^2 & C(-8,-4) \end{array}$$

We simplify the equations

$$\begin{array}{rrcl} 1) & k^2+h^2-4h+4 & = & r^2 \\ 2) & k^2+4k+4+h^2+4h+4 & = & r^2 \\ 3) & k^2+16k+64+h^2+8h+16 & = & r^2 \end{array} \qquad \begin{array}{rl} 1) & k^2+h^2-4h+4=r^2 \\ 2) & k^2+4k+h^2+4h+8=r^2 \\ 3) & k^2+16k+h^2+8h+80=r^2 \end{array}$$

We can eliminate r and all expressions quadratic in k if we subtract the first equation from the second and the third equations. We will proceed carefully, adding the opposite instead of subtracting.

$$\begin{array}{rrcl} 1) & -k^2-h^2+4h-4 & = & -r^2 \\ 2) & k^2+4k+h^2+4h+8 & = & r^2 \\ & & \Downarrow & \\ & 4k+8h+4 & = & 0 \\ & k+2h & = & -1 \end{array} \qquad \begin{array}{rr} 1) & -k^2-h^2+4h-4=-r^2 \\ 3) & k^2+16k+h^2+8h+80=r^2 \\ & \Downarrow \\ & 16k+12h+76=0 \\ & 4k+3h=-19 \end{array}$$

We obtained a system in two variables. Also, this system is now completely linear.

$$\begin{array}{rcl} k+2h & = & -1 \\ 4k+3h & = & -19 \end{array} \quad \text{multiply by } -4 \quad \Longrightarrow \quad \begin{array}{r} -4k-8h=4 \\ 4k+3h=-19 \end{array}$$

We add the two equations and obtain $-5h=-15$ and so $h=3$. From the equation $k+2h=-1$ we easily obtain the value $k=-7$. We now substitute $k=-7$, $h=3$ into the first equation to obtain the value of r^2.

$$\begin{array}{rcll} k^2+(2-h)^2 & = & r^2 & k=-7,\ h=3 \\ (-7)^2+(2-3)^2 & = & r^2 & \\ 50 & = & r^2 & \end{array}$$

And so the equation of the circle is $(x+7)^2+(y-3)^2=50$. It is interesting how the two solutions presented here relate to each other.

 Last revised: October 4, 2017