Completing the square is a powerful and elegant factoring technique. This technique is fundamentally different from other factoring techniques such as grouping or trial and error. While those other techniques enable us to factor over the integers, completing the square enables us to factor over the real numbers. If we wanted to factor expressions over the real numbers and the numbers turn out to be irrational, then grouping and trial and error will not work. Completing the square always does.

Completing the square is not just a factoring technique. We will soon see that it is also a way to understand quadratic expressions.

Part 1 - When the Leading Coefficient is 1

Definition: A **complete square** is an algebraic expression in which a sum (or difference) is being squared. For example, $(x-2)^2$, $(2a+3)^2$, and $(a+b-c)^2$ are complete squares.

Example 1. Expand each of the given complete squares.

a) $(x-2)^2$ b) $(2a+3)^2$

Solution: a) $(x-2)^2 = (x-2)(x-2) = x^2 - 2x - 2x + 4 = \boxed{x^2 - 4x + 4}$

b) $(2a+3)^2 = (2a+3)(2a+3) = 4a^2 + 6a + 6a + 9 = \boxed{4a^2 + 12a + 9}$

Example 2. Factor $-4x + x^2 - 21$ by completing the square.

Solution: Step 1. We re-arrange the terms by decreasing order of degree.

$$-4x + x^2 - 21 = x^2 - 4x - 21$$

Step 2A. We obtain the "magic number", that is half of the linear coefficient. The linear coefficient is the number multiplying x, **sign included**. In our example, the magic number is $\dfrac{-4}{2} = -2$. We do not write this line in the main computation.

Step 2B. We place an x in front of the magic number, and square the expression we obtained. Work out this computation on the margin, not in the main computation.

$$(x-2)^2 = (x-2)(x-2) = x^2 - 2x - 2x + 4 = x^2 - 4x + 4$$

Step 2C. We write the "helper line" $(x-2)^2 = x^2 - 4x + 4$ in the upper right hand side of the paper. We will use it twice. Our computation so far looks like this:

$$-4x + x^2 - 21 = x^2 - 4x - 21 \qquad\qquad (x-2)^2 = x^2 - 4x \boxed{+4}$$

Step 3. The smuggling step. What we have achieved in Step 2, is to have found the only perfect square that begins with the same two terms, $x^2 - 4x$ as our expression to be factored. We can see that the last term, $+4$ is missing. We complete the square as follows.

Step 3A. Write down our expression with one modification: we leave a gap between the second and third terms.

$$x^2 - 4x \qquad\quad - 21$$

Step 3B. We add zero to the expression by adding and then immediately subtracting 4 into the gap.

$$x^2 - 4x \ + 4 - 4 - 21$$

Step 4. We have obtained five terms. We re-write the first three terms as a perfect square (the second time we used the helper line) and combine the last two terms.

$$x^2 - 4x - 21 = \underbrace{x^2 - 4x + 4} - 4 - 21 = (x-2)^2 - 25$$

Step 5. We re-write the last number as a square.

$$(x-2)^2 - 25 = (x-2)^2 - 5^2$$

Step 6. If applies, we factor via the difference of squares theorem.

$$(x-2)^2 - 5^2 = (x-2+5)(x-2-5)$$

Step 7. (Cleanup) We simplify the factors by combining like terms.

$$(x-2+5)(x-2-5) = \boxed{(x+3)(x-7)}$$

Step 8. We check our result by multiplication.

$$(x+3)(x-7) = x^2 - 7x + 3x + 21 = x^2 - 4x + 21$$

Thus our result, $(x+3)(x-7)$ is correct.

The entire computation should look like this:

$$\begin{aligned}
-4x + x^2 - 21 &= \\
&= x^2 - 4x - 21 \qquad\qquad (x-2)^2 = x^2 - 4x + \boxed{4} \\
&= \underbrace{x^2 - 4x \ + 4} - 4 - 21 \\
&= (x-2)^2 - 25 \\
&= (x-2)^2 - 5^2 \\
&= (x-2+5)(x-2-5) \\
&= \boxed{(x+3)(x-7)}
\end{aligned}$$

We check: $(x+3)(x-7) = x^2 - 7x + 3x - 21 = x^2 - 4x - 21$

Example 3. Factor $32-18x+x^2$ by completing the square.

Solution: Step 1. We re-arrange the terms by decreasing order of degree.

$$32-18x+x^2=x^2-18x+32$$

Step 2A. We obtain the "magic number", that is half of the linear coefficient. (The linear coefficient is the number multiplying x, sign included.) In our example, the magic number is $\dfrac{-18}{2}=-9$.

Step 2B. We place an x in front of the magic number, and square the expression we obtained. Do not write this computation in the main computation.

$$(x-9)^2=(x-9)(x-9)=x^2-9x-9x+81=x^2-18x+81$$

Step 2C. We write the "helper line" $(x-9)^2=x^2-18x+81$ in the upper right hand side of the paper.

$$x^2-18x+32 \qquad\qquad (x-9)^2=x^2-18x+\boxed{81}$$

Step 3. (The smuggling step.) What we have achieved in Step 3, is to have found the only perfect square that begins with the same two terms, x^2-18x as our expression to be factored. We can see that the last term, $+81$ is missing. We complete the square by writing down our expression with a gap between the second and third terms, and then adding zero to the expression by adding and then immediately subtracting 81 in the gap.

$$x^2-18x+32=x^2-18x\ +81-81+32$$

Step 4. We obtained five terms. We re-write the first three terms as a perfect square and combine the last two terms.

$$x^2-18x+32=\underbrace{x^2-18x+81}-81+32=(x-9)^2-49$$

Step 5. We re-write the last number as a square.

$$(x-9)^2-49=(x-9)^2-7^2$$

Step 6. If applies, we factor via the difference of squares theorem.

$$(x-9)^2-7^2=(x-9+7)(x-9-7)$$

Step 7. (Cleanup) We simplify the factors by combining like terms.

$$(x-9+7)(x-9-7)=(x-2)(x-16)$$

Step 8. We check back by multiplication. (See below.)

The entire computation should look like this:

$$\begin{aligned}
32-18x+x^2 &= \\
&= x^2-18x+32 \qquad\qquad (x-9)^2=x^2-18x\ \boxed{+81}\\
&= \underbrace{x^2-18x\ +81}-81+32\\
&= (x-9)^2-49\\
&= (x-9)^2-7^2\\
&= (x-9+7)(x-9-7)\\
&= \boxed{(x-2)(x-16)}
\end{aligned}$$

We check: $(x-2)(x-16)=x^2-2x-16x+32=x^2-18x+32$

Thus our result, $(x-2)(x-16)$ is correct.

Example 4. Factor $28x + x^2 - 1173$ by completing the square.

Solution: We first rearrange the terms by degrees.

$$28x + x^2 - 1173 =$$

$$= x^2 + 28x - 1173 \qquad \text{half of the linear coefficient is } \frac{28}{2} = 14$$

We work out $(x+14)^2 = x^2 + 28x + 196$ on the margin. So we know to smuggle in 196.

$$(x+14)^2 = x^2 + 28x \boxed{+196}$$

$$\begin{aligned}
&= x^2 + 28x - 1173 && \\
&= \underbrace{x^2 + 28x + 196} - 196 - 1173 && \text{realize comple square, combine like terms} \\
&= (x+14)^2 - 1369 && \text{from calculator, } \sqrt{1369} = 37 \\
&= (x+14)^2 - 37^2 && \text{difference of squares theorem} \\
&= (x+14+37)(x+14-37) && \text{combine like terms} \\
&= \boxed{(x+51)(x-23)} &&
\end{aligned}$$

We check by multiplication: $(x+51)(x-23) = x^2 - 23x + 51x - 1173 = x^2 + 28x - 1173$

Thus our result, $(x+51)(x-23)$ is correct.

Example 5. Factor $17 - 2a + a^2$ by completing the square.

Solution: We first rearrange the terms by degree.

$$17 - 2a + a^2 =$$

$$\begin{aligned}
&= 17 - 2a + a^2 && \text{rearrange terms} \\
&= a^2 - 2a + 17 && \text{the "magic number" is } \frac{-2}{2} = -1
\end{aligned}$$

We work out $(a-1)^2 = a^2 - 2a + 1$ on the margin.

$$\begin{aligned}
&= a^2 - 2a + 17 && (a-1)^2 = a^2 - 2a \boxed{+1}\text{, so we smuggle in 1} \\
&= \underbrace{a^2 - 2a + 1} - 1 + 17 && \text{realize comple square, combine like terms} \\
&= (a-1)^2 + 16 &&
\end{aligned}$$

We can not apply the difference of squares theorem, since 16 is added, not subtracted. **The sum of squares can not be factored**, and so the expression $a^2 - 2a + 17$ can not be factored.

Practice Poblems

Factor each of the following by completing the square.

1. $x^2 - 10x + 21$
2. $x^2 - 6x + 8$
3. $22y + y^2 + 105$
4. $b^2 - 4b - 45$
5. $14a + a^2 - 51$
6. $b^2 - 10b + 26$
7. $3 + x^2 - 4x$
8. $d^2 + 2d + 2$
9. $6x + x^2 - 432$
10. $x^2 - 14x + 58$
11. $m^2 - 42m + 432$
12. $x^2 - 50x + 525$
13. $10y + y^2 - 375$
14. $x^2 - 40x + 336$
15. $x^2 - 6x + 25$
16. $q^2 - 2q - 48$
17. $x^2 - 18x + 81$
18. $t^2 - 36t - 4437$
19. $x^2 - 46x + 360$
20. $14q + q^2 - 2352$

Part 2 - When the Leading Coefficient is Not 1 but Can be Factored Out

Factoring by completing the square is an extremely powerful factoring technique. We will see later that this is the only method that does not break down once numbers stop being "nice". In this section, we will see expressions in which the leading coefficient is not 1, but it can be easily factored out.

Example 6. Factor $18x - 3x^2 + 165$ by completing the square.

Solution: Step 1. We re-arrange the terms by decreasing order of degree.

$$18x - 3x^2 + 165 = -3x^2 + 18x + 165$$

Step 2. We factor out the greatest common factor.

$$-3x^2 + 18x + 165 = -3\left(x^2 - 6x - 55\right)$$

Step 3. We factor the expression within the parentheses by completing the square.

$-3\left(x^2 - 6x - 55\right) =$ $\quad (x-3)^2 = x^2 - 6x + 9$

$$\begin{aligned} &= -3\left(\underbrace{x^2 - 6x + 9}\, - 9 - 55\right) \\ &= -3\left((x-3)^2 - 64\right) \\ &= -3\left((x-3)^2 - 8^2\right) \\ &= -3(x-3+8)(x-3-8) = \boxed{-3(x+5)(x-11)} \end{aligned}$$

Step 4. We check our result by multiplication.

$$-3(x+5)(x-11) = -3\left(x^2 - 11x + 5x - 55\right) = -3\left(x^2 - 6x - 55\right) = -3x^2 + 18x + 165$$

Thus our result, $-3(x+5)(x-11)$ is correct.

Example 7. Factor $267x^2 - 48x^3 + 3x^4$ by completing the square.

Solution: We first rearrange the terms by degree.

$267x^2 - 48x^3 + 3x^4 =$

$$\begin{aligned} &= 3x^4 - 48x^3 + 267x^2 && \text{factor out } 3x^2 \\ &= 3x^2\left(x^2 - 16x + 89\right) && (x-8)^2 = x^2 - 16x \boxed{+64} \\ &= 3x^2\left(\underbrace{x^2 - 16x + 64}\, - 64 + 89\right) && \text{realize comple square, combine like terms} \\ &= 3x^2\left((x-8)^2 + 25\right) \end{aligned}$$

We can not apply the difference of squares theorem, since 25 is added, not subtracted. The sum of squares cannot be factored. Therefore, the completely factored form of the expression is $\boxed{3x^2\left(x^2 - 16x + 89\right)}$

Example 8. Factor $5x^2 - 240x + 2160$ by completing the square.

Solution: We first factor out 5.

$$5x^2 - 240x + 2160 = 5\left(x^2 - 48x + 432\right) \qquad (x-24)^2 = x^2 - 48x \boxed{+576}$$

$$= 5\left(\underbrace{x^2 - 48x + 576} - 576 + 432\right)$$

$$= 5\left((x-24)^2 - 144\right)$$

$$= 5\left((x-24)^2 - 12^2\right)$$

$$= 5(x - 24 + 12)(x - 24 - 12)$$

$$= \boxed{5(x-12)(x-36)}$$

We check: $5(x-12)(x-36) = 5\left(x^2 - 12x - 36x + 432\right) = 5\left(x^2 - 48x + 432\right) = 5x^2 - 240x + 2160.$

Thus our result, $5(x-12)(x-36)$ is correct.

Example 9. Factor $9 - y^2 - 8y$ by completing the square.

Solution: We first rearrange the terms be degree and then factor out the leading coefficient.

$$9 - y^2 - 8y = -y^2 - 8y + 9$$

$$= -1\left(y^2 + 8y - 9\right) \qquad (y+4)^2 = y^2 + 8y \boxed{+16}$$

$$= -\left(\underbrace{y^2 + 8y + 16} - 16 - 9\right) \qquad \text{realize comple square, combine like terms}$$

$$= -\left((y+4)^2 - 25\right) \qquad \text{re-write 25 as a square}$$

$$= -\left((y+4)^2 - 5^2\right) \qquad \text{factor via the difference of squares theorem}$$

$$= -(y + 4 + 5)(y + 4 - 5) \qquad \text{combine like terms, drop extra parentheses}$$

$$= \boxed{-(y+9)(y-1)}$$

We check: $-(y+9)(y-1) = -\left(y^2 - y + 9y - 9\right) = -\left(y^2 + 8y - 9\right) = -y^2 - 8y + 9$

Thus our result, $-(y+9)(y-1)$ is correct.

Practice Poblems

Completely factor each of the following by completing the square.

1. $4x + 2x^2 - 30$
2. $70a^2 - 255a + 5a^3$
3. $78b^2 - 30b^3 + 3b^4$
4. $32x + 2x^2 - 594$
5. $18c - 24c^2 + 6c^3$
6. $-2d - 2d^2 - d^3$
7. $432 - x^2 - 6x$
8. $x^2 - 14x + 58$
9. $10abc - 600ac + 5ab^2c$
10. $70y^3 + 24y^4 + 2y^5$
11. $18x^2y^2 - 216x^2y + 3x^2y^3$
12. $1000x - 50x^2 - 5x^3$

Part 3 - With Fractions

It is strongly recommended to use fractions and not decimals. The steps are identical as before, the only complications are that we need to perform the same computations with fractions.

Example 10. Factor $54x - 6x^2 + 60$ by completing the square.

$$\begin{array}{rcl} 54x - 6x^2 + 60 & = & \text{rearrange terms} \\ -6x^2 + 54x + 60 & = & \text{factor out } -6 \\ -6\left(x^2 - 9x - 10\right) & = & \text{the "magic number" is } \frac{-9}{2} = -\frac{9}{2} \end{array}$$

We work out $\left(x - \frac{9}{2}\right)^2$ on the margin:

$$\left(x - \frac{9}{2}\right)^2 = \left(x - \frac{9}{2}\right)\left(x - \frac{9}{2}\right) = x^2 - \frac{9}{2}x - \frac{9}{2}x + \frac{81}{4} = x^2 - \frac{18}{2}x + \frac{81}{4} = x^2 - 9x + \frac{81}{4}$$

So we know to smuggle in the missing third term, $\frac{81}{4}$.

$$\text{our helper line: } \left(x - \frac{9}{2}\right)^2 = x^2 - 9x + \frac{81}{4}$$

$$\begin{array}{rll} -6\left(x^2 - 9x - 10\right) = & & \\ = & -6\left(\underbrace{x^2 - 9x + \frac{81}{4}} - \frac{81}{4} - \frac{10 \cdot 4}{1 \cdot 4}\right) & \text{realize complete square, re-write } 10 \text{ as } \frac{10}{1} = \frac{40}{4} \\ = & -6\left(\left(x - \frac{9}{2}\right)^2 - \frac{81}{4} - \frac{40}{4}\right) & \text{combine like terms} \\ = & -6\left(\left(x - \frac{9}{2}\right)^2 - \frac{121}{4}\right) & \text{re-write } \frac{121}{4} \text{ as } \left(\frac{11}{2}\right)^2 \\ = & -6\left(\left(x - \frac{9}{2}\right)^2 - \left(\frac{11}{2}\right)^2\right) & \text{factor via the difference of squares theorem} \\ = & -6\left(x - \frac{9}{2} + \frac{11}{2}\right)\left(x - \frac{9}{2} - \frac{11}{2}\right) & \text{combine like terms} \\ = & -6\left(x + \frac{2}{2}\right)\left(x - \frac{20}{2}\right) & \text{simplify fractions} \\ = & \boxed{-6\left(x+1\right)\left(x-10\right)} & \end{array}$$

We check by multiplication:

$$-6\left(x+1\right)\left(x-10\right) = -6\left(x^2 - 10x + x - 10\right) = -6\left(x^2 - 9x - 10\right) = -6x^2 + 54x + 60$$

Thus our result, $-6\left(x+1\right)\left(x-10\right)$ is correct.

Example 11. Factor $5p+p^2+6$ by completing the square.

$$\begin{aligned} 5p+p^2+6 &= \quad \text{rearrange terms} \\ p^2+5p+6 &= \quad \text{there is no GCF} \end{aligned}$$

The magic number is $\frac{5}{2}$. We work out $\left(p+\frac{5}{2}\right)^2$ on the margin:

$$\left(p+\frac{5}{2}\right)^2 = \left(p+\frac{5}{2}\right)\left(p+\frac{5}{2}\right) = p^2+\frac{5}{2}p+\frac{5}{2}p+\frac{25}{4} = p^2+\frac{10}{2}p+\frac{25}{4} = p^2+5p+\frac{25}{4}$$

So we know to smuggle in the missing third term, $\frac{25}{4}$.

$p^2+5p+6 =$ our helper line: $\left(p+\frac{5}{2}\right)^2 = p^2+5p+\frac{25}{4}$

$$\begin{aligned}
&= \underbrace{p^2+5p+\frac{25}{4}}-\frac{25}{4}+6 && \text{realize complete square} \\
&= \left(p+\frac{5}{2}\right)^2-\frac{25}{4}+\frac{6\cdot 4}{1\cdot 4} && \text{re-write } 6 \text{ as } \frac{6}{1}=\frac{24}{4} \\
&= \left(p+\frac{5}{2}\right)^2-\frac{25}{4}+\frac{24}{4} && \text{combine like terms} \\
&= \left(p+\frac{5}{2}\right)^2-\frac{1}{4} && \text{re-write } \frac{1}{4} \text{ as } \left(\frac{1}{2}\right)^2 \\
&= \left(p+\frac{5}{2}\right)^2-\left(\frac{1}{2}\right)^2 && \text{factor via the difference of squares theorem} \\
&= \left(p+\frac{5}{2}+\frac{1}{2}\right)\left(p+\frac{5}{2}-\frac{1}{2}\right) && \text{combine like terms} \\
&= \left(p+\frac{6}{2}\right)\left(p+\frac{4}{2}\right) && \text{simplify fractions} \\
&= \boxed{(p+3)(p+2)}
\end{aligned}$$

We check by multiplication:

$$(p+3)(p+2) = p^2+2p+3p+6 = p^2+5p+6$$

Thus our result, $(p+3)(p+2)$ is correct.

Note: These polynomials can easily be factored using trial and error and other methods. Why should we use completing the square? All other methods will break down once the numbers are less friendly. Then ONLY completing the square will work. Since those computations will be more difficult, you should learn this method while numbers are easy.

Example 12. Factor $3x^2 - 4x - 319$ by completing the square.

If the leading coefficient is not 1, we will factor it out before completing the square.

$$3x^2 - 4x - 319 = 3\left(x^2 - \frac{4}{3}x - \frac{319}{3}\right)$$

Half of the linear coefficient is $-\frac{4}{3} \div 2 = -\frac{4}{3} \cdot \frac{1}{2} = -\frac{4}{6} = -\frac{2}{3}$, thus we work out $\left(x - \frac{2}{3}\right)^2$.

$$\left(x - \frac{2}{3}\right)^2 = \left(x - \frac{2}{3}\right)\left(x - \frac{2}{3}\right) = x^2 - \frac{2}{3}x - \frac{2}{3}x + \frac{4}{9} = x^2 - \frac{4}{3}x + \frac{4}{9}$$

Thus we smuggle in $\frac{4}{9}$. The computation:

$$\begin{aligned}
&3x^2 - 4x - 319 = \\
&= 3\left(x^2 - \frac{4}{3}x - \frac{319}{3}\right) && \left(x - \frac{2}{3}\right)^2 = x^2 - \frac{4}{3}x + \frac{4}{9} \\
&= 3\left(\underbrace{x^2 - \frac{4}{3}x + \frac{4}{9}} - \frac{4}{9} - \frac{319 \cdot 3}{3 \cdot 3}\right) \\
&= 3\left(\left(x - \frac{2}{3}\right)^2 - \frac{4}{9} - \frac{957}{9}\right) \\
&= 3\left(\left(x - \frac{2}{3}\right)^2 - \frac{961}{9}\right) && \frac{961}{9} = \left(\frac{31}{3}\right)^2 \\
&= 3\left(\left(x - \frac{2}{3}\right)^2 - \left(\frac{31}{3}\right)^2\right) \\
&= 3\left(x - \frac{2}{3} + \frac{31}{3}\right)\left(x - \frac{2}{3} - \frac{31}{3}\right) \\
&= 3\left(x + \frac{29}{3}\right)\left(x - \frac{33}{3}\right) \\
&= 3\left(x + \frac{29}{3}\right)(x - 11) = \boxed{(3x + 29)(x - 11)}
\end{aligned}$$

We check: $(3x + 29)(x - 11) = 3x^2 - 33x + 29x - 319 = 3x^2 - 4x - 319$. Thus our answer is correct.

Example 13. Factor $11x+6x^2-10$ by completing the square.

We first rearrange the terms by degree and then factor out the leading coefficient.

$$11x+6x^2-10=6x^2+11x-10=6\left(x^2+\frac{11}{6}x-\frac{5}{3}\right)$$

Half of the linear coefficient is $\frac{11}{6}\div 2=\frac{11}{6}\left(\frac{1}{2}\right)=\frac{11}{12}$ and so we FOIL $\left(x+\frac{11}{12}\right)^2$.

$$\left(x+\frac{11}{12}\right)^2=\left(x+\frac{11}{12}\right)\left(x+\frac{11}{12}\right)=x^2+\frac{11}{12}x+\frac{11}{12}x+\left(\frac{11}{12}\right)^2=x^2+\frac{11}{6}x+\frac{121}{144} \qquad \frac{11}{12}+\frac{11}{12}=\frac{22}{12}=\frac{11}{6}$$

Thus we know to smuggle in $\frac{121}{144}$

$$
\begin{aligned}
6\left(x^2+\frac{11}{6}x-\frac{5}{3}\right) &= \\
&= 6\left(\underbrace{x^2+\frac{11}{6}x+\frac{121}{144}}-\frac{121}{144}-\frac{5}{3}\right) \qquad \frac{5}{3}=\frac{5\cdot 48}{3\cdot 48}=\frac{240}{144}\\
&= 6\left(\left(x+\frac{11}{12}\right)^2-\frac{121}{144}-\frac{240}{144}\right)\\
&= 6\left(\left(x+\frac{11}{12}\right)^2-\frac{361}{144}\right) \qquad \sqrt{361}=19 \text{ and } \sqrt{144}=12\\
&= 6\left(\left(x+\frac{11}{12}\right)^2-\left(\frac{19}{12}\right)^2\right)\\
&= 6\left(x+\frac{11}{12}+\frac{19}{12}\right)\left(x+\frac{11}{12}-\frac{19}{12}\right)\\
&= 6\left(x+\frac{30}{12}\right)\left(x-\frac{8}{12}\right)\\
&= 6\left(x+\frac{5}{2}\right)\left(x-\frac{2}{3}\right)=2\left(x+\frac{5}{2}\right)3\left(x-\frac{2}{3}\right)=\boxed{(2x+5)(3x-2)}
\end{aligned}
$$

We check: $(2x+5)(3x-2)=6x^2-4x+15x-10=6x^2+11x-10$. Thus our answer, $(2x+5)(3x-2)$ is correct.

Practice Poblems

Completely factor each of the following by completing the square.

1. $x+x^2-12$
2. x^2-x-90
3. $11x+x^2+30$
4. $x^2-17x+72$
5. $3x+3x^2-60$
6. $2x^2-6x+17$
7. $x+2x^2-1$
8. $112x+2x^2-2x^3$
9. $5a^2-14a-3$
10. $33c^4-270c^3-c^5$
11. $m+6m^2-2$
12. $6x^2-7x-3$
13. $10p^2-11p-6$
14. $15x^2-34x+15$

Enrichment

Recall the difference of cubes theorem: For any quantities A and B, $A^3 - B^3$ can be factored as:

$$A^3 - B^3 = (A - B)\left(A^2 + AB + B^2\right).$$

1. Verify this identity by expanding the right-hand side.
2. Expand the given products. What do you notice?
   $(3x - 2)(5x + 1)$ and $(3A - 2B)(5A + B)$
3. Consider now the second, quadratic factor in $A^3 - B^3 = (A - B)\left(A^2 + AB + B^2\right)$. Suppose we treat A as if it was x, and B as if it was 1. What happens when we try to factor $A^2 + AB + B^2$ by completing the square?
4. What conclusions, if any, can we draw from our result?

Answers

Practice Problems - Part 1

1. $(x-3)(x-7)$ 2. $(x-2)(x-4)$ 3. $(y+15)(y+7)$ 4. $(b+5)(b-9)$ 5. $(a+17)(a-3)$

6. can not be factored 7. $(x-1)(x-3)$ 8. can not be factored 9. $(x+24)(x-18)$

10. can not be factored 11. $(m-18)(m-24)$ 12. $(x-15)(x-35)$ 13. $(y+25)(y-15)$

14. $(x-12)(x-28)$ 15. can not be factored 16. $(q+6)(q-8)$ 17. $(x-9)^2$

18. $(t+51)(t-87)$ 19. $(x-10)(x-36)$ 20. $(q+56)(q-42)$

Practice Problems - Part 2

1. $2(x+5)(x-3)$ 2. $5a(a-3)(a+17)$ 3. $3b^2\left(b^2-10b+26\right)$ 4. $2(x+27)(x-11)$

5. $6c(c-1)(c-3)$ 6. $-d\left(d^2+2d+2\right)$ 7. $-(x+24)(x-18)$ 8. can not be factored

9. $5ac(b+12)(b-10)$ 10. $2y^3(y+7)(y+5)$ 11. $3x^2y(y+12)(y-6)$ 12. $-5x(x+20)(x-10)$

Practice Problems - Part 3

1. $(x+4)(x-3)$ 2. $(x+9)(x-10)$ 3. $(x+6)(x+5)$ 4. $(x-8)(x-9)$ 5. $3(x+5)(x-4)$

6. can not be factored 7. $2(x+1)\left(x-\frac{1}{2}\right) = (x+1)(2x-1)$ 8. $-2x(x+7)(x-8)$

9. $5\left(a+\frac{1}{5}\right)(a-3) = (5a+1)(a-3)$ 10. $-c^3(c-15)(c-18)$

11. $6\left(m+\frac{2}{3}\right)\left(m-\frac{1}{2}\right) = (3m+2)(2m-1)$ 12. $6\left(x+\frac{1}{3}\right)\left(x-\frac{3}{2}\right) = (3x+1)(2x-3)$

13. $10\left(p+\frac{2}{5}\right)\left(p-\frac{3}{2}\right) = (5p+2)(2p-3)$ 14. $15\left(x-\frac{3}{5}\right)\left(x-\frac{5}{3}\right) = (5x-3)(3x-5)$

Enrichment

If we treat A as our variable x, and B like the numbers inthe previous example. Then A^2 is quadratic, AB is linear, and B^2 is the constant term. We proceed to complete the square. Half of the linear coefficient is $\frac{1}{2}B$.

So the complete square we need is $\left(A+\frac{B}{2}\right)^2$

$$\left(A+\frac{B}{2}\right)^2=A^2+\frac{AB}{2}+\frac{AB}{2}+\frac{B^2}{4}=A^2+AB+\frac{B^2}{4}$$

So we will smuggle in $\frac{B^2}{4}$

$$\begin{aligned}
A^2+AB+B^2 &= \\
A^2+AB+\frac{B^2}{4}-\frac{B^2}{4}+B^2 &= \\
\left(A+\frac{B}{2}\right)^2-\frac{B^2}{4}+\frac{4B^2}{4} &= \left(A+\frac{B}{2}\right)^2+\frac{3}{4}B^2 \\
&= \left(A+\frac{B}{2}\right)^2+\left(\frac{\sqrt{3}}{2}B\right)^2
\end{aligned}$$

Since the sum of two squares can not be factored, the expression A^2+AB+B^2 is irreducible, can not be factored.

Conclusons:

If the square of two quantities are the same, that does not mean that the quantities are the same. They also could be opposites.

$$\begin{aligned}
A^2 &= B^2 \\
A^2-B^2 &= 0 \\
(A+B)(A-B) &= 0 \implies A=B \text{ or } A=-B
\end{aligned}$$

But if the cube of two quantities are the same, they must equal. This is an interesting difference.

$$\begin{aligned}
A^3 &= B^3 \\
A^3-B^3 &= 0 \\
(A-B)\left(A^2+AB+B^2\right) &= 0 \implies A=B
\end{aligned}$$

We have just proved that the second, longer factor A^2+AB+B^2 is a sum of two squares, and therefore is always positive. Therefore, this factor does not produce any solutions. So the only solution of $A^3=B^3$ is $A=B$, while the solutions of $A^2=B^2$ are $A=\pm B$.

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