Part 1 - Definitions

Equations are a fundamental concept and tool in mathematics.

> **Definition**: An **equation** is a statement in which two expressions (algebraic or numeric) are connected with an equal sign.

For example, $3x^2 - x = 4x + 28$ is an equation. So is $x^2 + 5y = -y^2 + x + 2$.

> **Definition**: A **solution** of an equation is a number (or an ordered set of numbers) that, when substituted into the variable(s) in the equation, makes the statement of equality true.

Example 1. a) Verify that -2 is not a solution of the equation $3x^2 - x = 4x + 28$.

b) Verify that 4 is a solution of the equation $3x^2 - x = 4x + 28$.

c) Given the values -4,-2, and 6, find all numbers listed that are solution of the equation $x^3 - x^2 - 27x + 2 = x^2 - 3x + 2$. Solution - Youtube link

Solution: a) Consider the equation $3x^2 - x = 4x + 28$ with $x = -2$. We substitute $x = -2$ into both sides of the equation and evaluate the expressions.

If $x = -2$, the left-hand side of the equation is $\text{LHS} = 3x^2 - x = 3(-2)^2 - (-2) = 3 \cdot 4 + 2 = 12 + 2 = 14$	If $x = -2$, the right-hand side of the equation is $\text{RHS} = 4x + 28 = 4(-2) + 28 = -8 + 28 = 20$

Since the two sides are not equal, $14 \neq 20$, the number -2 is not a solution of this equation.

b) Consider the equation $3x^2 - x = 4x + 28$ with $x = 4$. We evaluate both sides of the equation after substituting 4 into x.

If $x = 4$, the left-hand side of the equation is $\text{LHS} = 3x^2 - x = 3 \cdot 4^2 - 4 = 3 \cdot 16 - 4 = 48 - 4 = 44$	If $x = 4$, the right-hand side of the equation is $\text{RHS} = 4x + 28 = 4 \cdot 4 + 28 = 16 + 28 = 44$

Since the two sides are equal, $x = 4$ is a solution of this equation.

> **Definition**: To **solve an equation** is to find *all* solutions of it. The set of all solutions is also called the solution set.

Caution! Finding one solution for an equation is not the same as solving it. For example, the number 2 is a solution of the equation $x^3 = 4x$. However, -2 is also a solution of this equation.

Example 2: Consider the equation $x^2 - y^2 = 2y - x + 3$.

a) Verify that the ordered pair $(1, -3)$ is not a solution of the equation.

b) Verify that the ordered pair $(-3, 1)$ is a solution of the equation.

Solution: *Ordered pair* simply means that the order of the two numbers listed matters. $(1, -3)$ indicates that $x = 1$ and $y = -3$. On the other hand, the ordered pair $(-3, 1)$ means that $x = -3$ and $y = 1$. Let us see if either pair is a solution.

a) Checking the ordered pair $(1, -3)$ - this means that $x = 1$ and $y = -3$. We substitute these values into both sides of the equation $x^2 - y^2 = 2y - x + 3$.

$$\begin{aligned} \text{the left-hand side is} \quad \text{LHS} &= 1^2 - (-3)^2 = 1 - 9 = -8 \\ \text{the right-hand side is} \quad \text{RHS} &= 2(-3) - 1 + 3 = -6 - 1 + 3 = -7 + 3 = -4 \\ -8 &\neq -4 \\ \text{LHS} &\neq \text{RHS} \end{aligned}$$

The ordered pair $(1, -3)$ is not a solution of this equation.

b) Checking the ordered pair $(-3, 1)$ - this means that $x = -3$ and $y = 1$. We substitute these values into both sides of the equation $x^2 - y^2 = 2y - x + 3$.

$$\begin{aligned} \text{the left-hand side is} \quad \text{LHS} &= (-3)^2 - 1^2 = 9 - 1 = 8 \\ \text{the right-hand side is} \quad \text{RHS} &= 2 \cdot 1 - (-3) + 3 = 2 + 3 + 3 = 8 \\ 8 &= 8 \\ \text{LHS} &= \text{RHS} \end{aligned}$$

The ordered pair $(-3, 1)$ is a solution of this equation.

Definition: To **solve an equation** is to find *all* solutions of it. The set of all solutions is also called the solution set.

Caution! Finding one solution for an equation is not the same as solving it. For example, we found that -5 is a solution of $-x^2 + 3 = 4x - 2$. As it turns out, -5 is not the only solution. We leave to the reader to verify that 1 is also a solution of the equation.

If we think about it a little, trial and error could rarely be a legitimate method because there is no way for us to gurantee that there are no other solutions are. It is impossible for us to try all real numbers because there are infinitely many of them, and we have finite lives.

So we will need to develop systematic methods to solve equations. We will start with the easiest group of equations, linear equations. There are several types of linear equations, and we will start with the simplest types of linear equations.

Part 2 - One- and Two-Step Equations

If you need more resources, please check out the handout Solving One- and Two-Step Linear Equations.

Example 3v. Solve each of the given equations. Make sure to check your solutions.

a) $3x + 8 = -7$ Solution - Youtube link

b) $-5x + 3 = 38$ Solution - Youtube link

c) $\dfrac{x+20}{3} = 2$ Solution - Youtube link

d) $\dfrac{x+3}{-5} = 2$ Solution - Youtube link

e) $\dfrac{x-2}{5} = -1$ Solution - Youtube link

f) $\dfrac{x}{-4} + 5 = 8$ Solution - Youtube link

Example 4. The sum of three times a number and seven is -5. Find this number.

Solution: Let us denote our mystery number by x. The equation will be just the first sentence, translated to algebra. The sum of three times the number and seven is $3x+7$. So our equation is $3x+7=-5$. We will know the number if we solve this equation.

$$\begin{aligned} 3x + 7 &= -5 && \text{subtract } 7 \\ 3x &= -12 && \text{divide by } 3 \\ x &= -4 \end{aligned}$$

Good news! We do not need to check if -4 is indeed the solution of the equation. What if we correctly solved the *wrong* equation? Recall that *we* came up with the equation, it was not given. Instead of checking the number against the equation, we should check if our solution satisfies the conditions stated in the problem. Is it true the sum of three times -4 and seven is -5? Indeed,
$3(-4) + 7 = -12 + 7 = -5$. Thus our solution, $\boxed{-4}$, is correct.

Example 4v. Translate the given sentence to an equation using x for the number. Then solve the equation for x.

a) The sum of a number and four is ten. Find this number. Solution - Youtube link

b) The sum of three times a number and six is nine. Find this number. Solution - Youtube link

c) The product of three more than a number and eight is 48. Find this number. Solution - Youtube link

Example 4w. Susan is asked about her age. She answers as follows. "*My age is 8 years less than twice the age of my brother.*" How old is her brother if Susan is 24 years old? Solution - Youtube link

Example 5v. The cost of a taxi ride is $\$1.55$ for the first mile and $\$1.35$ for each additional mile or part thereof. Find the distance we covered if we paid $\$24.50$. Solution - Youtube link

Example 6. Solve each of the following equations.

a) $2x-3=15$ b) $\dfrac{2}{3}x-\dfrac{1}{2}=-\dfrac{1}{3}$ c) $\dfrac{x-4}{3}=-6$ d) $\dfrac{x+\frac{1}{2}}{\frac{3}{5}}=\dfrac{1}{3}$

Solution: a) There is nothing new or unusual about this equation. The right-hand side is just a number. The unknown only appears on the left-hand side. There, we see two operations: the unknown was first multiplied by 2 and then 3 was subtracted. To isolate the unknown on the left-hand side, we will perform the inverse operations to both sides, in a reverse order. We will first add 3 and then we will divide by 2.

$$\begin{aligned} 2x-3 &= 15 && \text{add } 3 \\ 2x &= 18 && \text{divide by } 2 \\ x &= 9 \end{aligned}$$

We check: If $x=9$, then the left-hand side is:

LHS $=2x-3=2\cdot 9-3=18-3=15=$ RHS ✓

Thus our solution, $x=\boxed{9}$ is correct.

b) Consider the equation $\dfrac{2}{3}x-\dfrac{1}{2}=-\dfrac{1}{3}$. There is nothing new or unusual about this equation either. If we could solve $2x-3=15$, then we can solve this equation using the same steps. On the left-hand side, the unknown was multiplied by $\dfrac{2}{3}$ and then $\dfrac{1}{2}$ was subtracted. To isolate the unknown, we will perform to both sides the inverse operations, in a reverse order. This means that we will add $\dfrac{1}{2}$ and then divide by $\dfrac{2}{3}$. The main computation should be clean; we should just record the result of each step. We will perform computations on the margin.

$$\begin{aligned} \frac{2}{3}x-\frac{1}{2} &= -\frac{1}{3} && \text{add } \frac{1}{2} \\ \frac{2}{3}x &= \frac{1}{6} && \text{divide by } \frac{2}{3} \\ x &= \frac{1}{4} \end{aligned}$$

margin work: $-\dfrac{1}{3}+\dfrac{1}{2}=\dfrac{-2}{6}+\dfrac{3}{6}=\dfrac{1}{6}$

$\dfrac{1}{6}\div\dfrac{2}{3}=\dfrac{1}{6}\cdot\dfrac{3}{2}=\dfrac{1}{2\cdot \cancel{3}}\cdot\dfrac{\cancel{3}}{2}=\dfrac{1}{4}$

We check: If $x=\dfrac{1}{4}$, then the left-hand side is

LHS $=\dfrac{2}{3}\left(\dfrac{1}{4}\right)-\dfrac{1}{2}=\dfrac{\cancel{2}}{3}\cdot\dfrac{1}{\cancel{2}\cdot 2}-\dfrac{1}{2}=\dfrac{1}{6}-\dfrac{1}{2}=\dfrac{1}{6}-\dfrac{3}{6}=\dfrac{-2}{6}=-\dfrac{1}{3}=$ RHS ✓

Thus our solution, $x=\boxed{\dfrac{1}{4}}$ is correct.

c) Consider now the equation $\dfrac{x-4}{3}=-6$. This is a simple two-step equation, we only have it here to serve as an analogous example for part d. On the left-hand side we have first a subtraction of 4 and then a division by 3. To isolate the unknown, we will multiply by 3 and then add 4. As always, we wil perform all operations to both sides.

$$\begin{aligned}\frac{x-4}{3} &= -6 \qquad \text{multiply by } 3\\ x-4 &= -18 \qquad \text{add } 4\\ x &= -14\end{aligned}$$

We check: If $x=-14$, then the left-hand side is:
$\text{LHS} = \dfrac{x-4}{3} = \dfrac{-14-4}{3} = \dfrac{-18}{3} = -6 = \text{RHS}$ ✓
Thus our solution, $x = \boxed{-14}$ is correct.

d) Consider now the equation $\dfrac{x+\frac{1}{2}}{\frac{3}{5}} = \dfrac{1}{3}$. If we could solve $\dfrac{x-4}{3}=-6$, then we can solve this equation using the same steps. On the left-hand side, there was an addition of $\dfrac{1}{2}$, and then a division by $\dfrac{3}{5}$. To isolate the unknown, we will perform to both sides the inverse operations, in a reverse order. This means that we will multiply by $\dfrac{3}{5}$ and then subtract $\dfrac{1}{2}$. The main computation should be clean; we should just record the result of each step. We will perform computations on the margin.

$$\begin{aligned}\frac{x+\frac{1}{2}}{\frac{3}{5}} &= \frac{1}{3} \qquad \text{multiply by } \frac{3}{5}\\ x+\frac{1}{2} &= \frac{1}{5} \qquad \text{subtract } \frac{1}{2}\\ x &= -\frac{3}{10}\end{aligned}$$

margin work: $\dfrac{1}{3}\cdot\dfrac{3}{5}=\dfrac{1}{5}$

$\dfrac{1}{5}-\dfrac{1}{2}=\dfrac{2-5}{10}=-\dfrac{3}{10}$

We check: If $x=-\dfrac{3}{10}$, then the left-hand side is

$$\text{LHS} = \frac{x+\frac{1}{2}}{\frac{3}{5}} = \frac{-\frac{3}{10}+\frac{1}{2}}{\frac{3}{5}} = \frac{\frac{-3+5}{10}}{\frac{3}{5}} = \frac{\frac{2}{10}}{\frac{3}{5}} = \frac{\frac{1}{5}}{\frac{3}{5}} = \frac{1}{5}\cdot\frac{5}{3} = \frac{1}{3} = \text{RHS}$$

Thus our solution, $x = \boxed{-\dfrac{3}{10}}$ is correct.

The next example is not a two-step equation but it can be solved using the same method. We will see many steps. We will perform the inverse operations, in the reverse order. It is sort of like an onion we take apart. We can always peel off the outermost layer.

Example 7. Solve the given equations

a) . $\dfrac{\dfrac{\dfrac{3x-1}{5}+2}{-2}-8}{4}=-2$ b) $\dfrac{\dfrac{x-5}{-4}+5}{4}-1=4$ Solution - Youtube link

Solution: a) The unknown is on the left-hand side, and lots of operations were done to it. In order, there were: multiplication by 3, subtracting 1, division by 5, adding 2, division by -2, subtraction of 8 and division by 4. We will undo them in the reversed order. This means: first multiply by 4, then add 8, then multiply by -2, then subtract 2, then multiply by 5, then add 1, and finally divide by 3. So, that's the plan.

$$\dfrac{\dfrac{\dfrac{3x-1}{5}+2}{-2}-8}{4} = -2 \quad \text{multiply by } 4$$

$$\dfrac{\dfrac{3x-1}{5}+2}{-2}-8 = -8 \quad \text{add } 8$$

$$\dfrac{\dfrac{3x-1}{5}+2}{-2} = 0 \quad \text{multiply by } -2$$

$$\dfrac{3x-1}{5}+2 = 0 \quad \text{subtract } 2$$

$$\dfrac{3x-1}{5} = -2 \quad \text{multiply by } 5$$

$$3x-1=-10 \quad \text{add } 1$$

$$3x=-9 \quad \text{divide by } 3$$

$$x=-3$$

We check: if $x=-3$, then the left-hand side is

$$\text{LHS} = \dfrac{\dfrac{\dfrac{3(-3)-1}{5}+2}{-2}-8}{4} = \dfrac{\dfrac{\dfrac{-9-1}{5}+2}{-2}-8}{4} = \dfrac{\dfrac{\dfrac{-10}{5}+2}{-2}-8}{4} = \dfrac{\dfrac{-2+2}{-2}-8}{4} = \dfrac{\dfrac{0}{-2}-8}{4}$$

$$= \dfrac{0-8}{4} = \dfrac{-8}{4} = -2 = \text{RHS} \checkmark$$

Thus our solution, $\boxed{x=-3}$ is correct.

Part 3 - Linear Equations with More Algebra

Example 8. Solve each of the given equations. Make sure to check your solutions.

a) $2x-8=5x+10$ b) $7a-12=-a+20$ c) $-4x+2=-x+17$ d) $\frac{1}{2}m-1=\frac{5}{4}m-\frac{1}{4}$

Solution: Notice that in each equation, the unknown appears on both sides. This will be the first thing we will address.

a)

$$\begin{aligned} 2x-8 &= 5x+10 && \text{subtract } 2x \\ -8 &= 3x+10 && \text{subtract } 10 \\ -18 &= 3x && \text{divide by } 3 \\ -6 &= x \end{aligned}$$

So the only solution of this equation is -6. We check; if $x=-6$,

$$\text{LHS} = 2(-6)-8 = -12-8 = -20 \text{ and } \text{RHS} = 5(-6)+10 = -30+10 = -20 \implies \text{LHS} = \text{RHS}$$

So our solution, $\boxed{x=-6}$ is correct.

b)

$$\begin{aligned} 7a-12 &= -a+20 && \text{add } a \\ 8a-12 &= 20 && \text{add } 12 \\ 8a &= 32 && \text{divide by } 8 \\ a &= 4 \end{aligned}$$

So the only solution of this equation is 4. We check; if $a=4$,

$$\text{LHS} = 7\cdot 4-12 = 28-12 = 16 \text{ and } \text{RHS} = -4+20 = 16 \implies \text{LHS} = \text{RHS}$$

So our solution, $\boxed{a=4}$ is correct.

c)

$$\begin{aligned} -4x+2 &= -x+17 && \text{add } 4x \\ 2 &= 3x+17 && \text{subtract } 17 \\ -15 &= 3x && \text{divide by } 3 \\ -5 &= x \end{aligned}$$

We check; if $x=-5$, then

$$\text{LHS} = -4(-5)+2 = 20+2 = 22 \text{ and } \text{RHS} = -(-5)+17 = 5+17 = 22 \implies \text{LHS} = \text{RHS}$$

So our solution, $\boxed{x=-5}$ is correct.

d) $\frac{1}{2}m-1=\frac{5}{4}m-\frac{1}{4}$ subtract $\frac{1}{2}m$ margin work: $\frac{5}{4}-\frac{1}{2}=\frac{5}{4}-\frac{2}{4}=\frac{3}{4}$

$-1=\frac{3}{4}m-\frac{1}{4}$ add $\frac{1}{4}$ $-1+\frac{1}{4}=\frac{-4}{4}+\frac{1}{4}=-\frac{3}{4}$

$-\frac{3}{4}=\frac{3}{4}m$ divide by $\frac{3}{4}$ $-\frac{3}{4}\div\frac{3}{4}=-\frac{3}{4}\cdot\frac{4}{3}=-1$

$-1=m$

So the only solution of this equation is -1. We check; if $m=-1$,

$$\text{LHS}=\frac{1}{2}(-1)-1=-\frac{1}{2}-1=\frac{-1}{2}-\frac{2}{2}=-\frac{3}{2}\quad\text{and}$$

$$\text{RHS}=\frac{5}{4}(-1)-\frac{1}{4}=-\frac{5}{4}-\frac{1}{4}=-\frac{6}{4}=-\frac{3}{2}\qquad\Longrightarrow\qquad\text{LHS}=\text{RHS}$$

So our solution, $\boxed{m=-1}$ is correct.

Linear equations might be more complicated. Most often we will be dealing with the distributive law. Also, these equations can be classified based on their solution sets. Consider each of the following.

Example 9. Solve each of the given equations. Make sure to check your solutions.

a) $3x-2(4-x)=3(3x-1)-(x-7)$

b) $4(y-2)-6(3y-5)=5-2(7y+1)$

c) $\frac{2}{3}x-4-\frac{1}{6}(x+6)=\frac{1}{2}(x-10)$

d) $4(3x+1)-(x+3)=10x-4$
Solution - YouTube Link

Solution: a) We first eliminate the parentheses by applying the distributive law.

$$\begin{aligned}
3x-2(4-x) &= 3(3x-1)-(x-7) && \text{eliminate parentheses} \\
3x-8+2x &= 9x-3-x+7 && \text{combine like terms} \\
5x-8 &= 8x+4 && \text{subtract } 5x \\
-8 &= 3x+4 && \text{subtract } 4 \\
-12 &= 3x && \text{divide by } 3 \\
-4 &= x
\end{aligned}$$

Caution! $-2(-x)=2x$ and $-(-7)=7$

We check: if $x=-4$, then

$$\begin{aligned}
\text{LHS} &= 3(-4)-2(4-(-4))=3(-4)-2\cdot 8=-12-16=-28 \quad\text{and} \\
\text{RHS} &= 3(3(-4)-1)-(-4-7)=3(-12-1)-(-11)=3(-13)+11=-39+11=-28
\end{aligned}$$

$\Longrightarrow$ LHS = RHS

So our solution, $\boxed{x=-4}$ is correct.

b) We first eliminate the parentheses by applying the distributive law.

$$\begin{aligned} 4(y-2)-6(3y-5) &= 5-2(7y+1) && \text{eliminate parentheses} \\ 4y-8-18y+30 &= 5-14y-2 && \text{combine like terms} \\ -14y+22 &= -14y+3 && \text{add } 14y \\ 22 &= 3 \end{aligned}$$

Something different happened here. When we tried to eliminate the unknown from one side, it disappeared from both sides. We are left with the statement $22 = 3$. No matter what the value of the unknown is, this statement can not be made true. Indeed, our last line is an **unconditionally false statement**. This means that there is no number that could make this statement true, and so this equation **has no solution**. An equation like this is called a **contradiction**.

c) We first eliminate the parentheses by applying the distributive law.

$$\begin{aligned} \frac{2}{3}x-4-\frac{1}{6}(x+6) &= \frac{1}{2}(x-10) && \text{eliminate parentheses} \\ \frac{2}{3}x-4-\frac{1}{6}x-1 &= \frac{1}{2}x-5 && \text{combine like terms} \quad \text{margin work: } \frac{2}{3}-\frac{1}{6}=\frac{4-1}{6}=\frac{3}{6}=\frac{1}{2} \\ \frac{1}{2}x-5 &= \frac{1}{2}x-5 && \text{subtract } \frac{1}{2}x \\ -5 &= -5 \end{aligned}$$

When we tried to eliminate the unknown from one side, it disappeared again from both sides. We are left with the statement $-5 = -5$. No matter what the value of the unknown is, this statement is always true. Indeed, our last line is an **unconditionally true statement**. This means that every number makes make this statement true, and so the solution set of this equation is the set of all numbers. An equation like this is called an **identity**.

We often use identities in mathematics, although it seems at first that we would not need equations whose solution set is every number. Consider the following equation: $a+b = b+a$. This equation is an identity, because every pair of numbers is a solution. We use this identity to express a property of *addition*: that the sum of two numbers does not depend on the order of the two numbers.

Based on their solution sets, these equations can be classified as belonging to one of the following three groups.

1. If the last line is of the form $x = 5$, the equation is called **conditional**. (This is because the truth value of the statement depends on the value of x. True if x is 5, false otherwise.) A conditional equation has exactly one solution.

2. If the last line is of the form $1 = 1$, the equation is unconditionally true. Such an equation is called an **identity** and all numbers are solutions of it.

3. If the last line is of the form $3 = 14$, the equation is unconditionally false. Such an equation is called a **contradiction** its solution set is the empty set.

Discussion: Classify each of the following equations as conditional, identity, or contradiction.
a) $3x+1=3x-1$ b) $2x-4=7x-4$ c) $x-4=4-x$ d) $x-1=-1+x$

Example 10. Ann was asked about her age. Her answer was: "*My age is 10 years less than three times my brother's age. The sum of our ages is* 18 *years*". How old are the siblings?

Solution: Let us denote the brother's age by x. Then Ann's age can be expressed as $3x-10$. The equation will express the sum of these two quantities.

$$\begin{aligned} x+3x-10 &= 18 && \text{combine like terms} \\ 4x-10 &= 18 && \text{add } 10 \\ 4x &= 28 && \text{divide by } 4 \\ x &= 7 \end{aligned}$$

So, if the age of Ann's brother is 7 years, then Ann's age is $3\cdot 7-10=11$ years old. So the siblings are $\boxed{7 \text{ and } 11 \text{ years}}$ old. We check: the ages add up to 18, and 11 is ten less than three times 7.

Example 10v. Wendy is asked about her age. She answers as follows. "My age is seven years less than twice the age of my son. The sum of our ages is 61 years. How old is Wendy?" **Solution - YouTube Link**

Example 11. Three times the sum of a number and two is fifteen greater than the sum of one and the number. Find this number.

Solution: Let us denote the number by x. Then thre times the sum of this number and two can be translated as $3(x+2)$. The sum of one and the number is $1+x$. The equation will express the comparison between the two.

$$\begin{aligned} 3(x+2) &= 1+x+15 && \text{distribute 3 and combine like terms} \\ 3x+6 &= x+16 && \text{subtract } x \\ 2x+6 &= 16 && \text{subtract } 6 \\ 2x &= 10 && \text{divide by } 2 \\ x &= 5 \end{aligned}$$

Therefore, this number is $\boxed{5}$. We check: Three times the sum of our number and two is $3\cdot 7=21$, and the sum of one and the number is 6. Indeed, 21 is fifteen greater than 6, so our solution is correct.

Example 11v. Five times the sum of four and a number is six less than seven times the same number.

Solution - YouTube Link

Example 12. Children's tickets cost 12 dollars, and adult tickets cost 20 dollars. The number of children's tickets purchased was five more than three times the number of adult tickets purchased. How many of each tickets did we buy if we paid 900 dollars for all these tickets?

Solution: Let us denote the number of adult tickets purchased by x. Then the number of children's ticket can be expressed as $3x+5$. We obtain an equation expressing the value of all the tickets.

We paid 20 dollars for each of the x many adult tickets. So the adult tickets cost. $20x$ dollars. Similarly, each children's ticket cost 12 dollars, and we purchased $3x+5$ many children's tickets. Therefore, we paid $12(3x+5)$ for the children's tickets. We will now set up an equation to express the total value of all tickets:

$$\begin{aligned} 20x+12(3x+5) &= 900 && \text{apply the distributive law} \\ 20x+36x+60 &= 900 && \text{combine like terms} \\ 56x+60 &= 900 && \text{subtract 60} \\ 56x &= 840 && \text{divide by 56} \\ x &= 15 \end{aligned}$$

Since x represents the number of adult tickets, our result is that 15 adult tickets were purchased. Then the children's ticket, denoted by $3x+5$, must be $3\cdot 15+5=50$. So it appears that the correct answer is $\boxed{\text{15 adults and 50 children tickets}}$.

We check: $3\cdot 15+5=50$ and so the number of children's ticket is indeed 5 more than three times the number of adult tickets. As for the total cost, $12(50)+15(20)=600+300=900$. This means that our solution is correct.

Example 12v. The tickets for the field trip were purchased yesterday for both students and instructors. Children tickets cost 11 dollars, adult tickets cost 14 dollars. The number of children tickets purchased was five less than twice the number of adults tickets purchased. How many of each were purchased if all of the tickets cost a total of 465 dollars? **Solution - YouTube Link**

Example 13. Solve each of the given equations. Make sure to check your solutions.

a) $2(8x-5(2x+1))-(x+1)=49$

b) $-3(x+4(x+3(-3x-1)))=-3+7(8x-5)$

c) $-5(x+2(x-2(-4x+3)))=60(-5x+1)$

a) We simplify the expression on the right-hand side. In this case, this might take a litlle bit more steps. After that, the problem is reduced to a two-step equation.

$$\begin{aligned} 2(8x-5(2x+1))-(x+1) &= 49 && \text{apply distributive law to open the inner parentheses} \\ 2(8x-10x-5)-(x+1) &= 49 && \text{combine like terms} \\ 2(-2x-5)-(x+1) &= 49 && \text{apply distributive law again} \\ -4x-10-x-1 &= 49 && \text{combine like terms again} \\ -5x-11 &= 49 && \text{add 11} \\ -5x &= 60 && \text{divide by } -5 \\ x &= -12 \end{aligned}$$

We check: if $x=-12$, then

$$\begin{aligned}\text{LHS} &= 2\left(8\left(-12\right)-5\left(2\left(-12\right)+1\right)\right)-\left(\left(-12\right)+1\right)=2\left(-96-5\left(-24+1\right)\right)-\left(-11\right)\\ &= 2\left(-96-5\left(-23\right)\right)+11=2\left(-96+115\right)+11=2\left(19\right)+11=38+11=49=\text{RHS}\ \checkmark\end{aligned}$$

So $x=\boxed{-12}$ is our solution.

b) $-3\left(x+4\left(x+3\left(-3x-1\right)\right)\right)=-3+7\left(8x-5\right)$ **Solution - Youtube link**

c) $-5\left(x+2\left(x-2\left(-4x+3\right)\right)\right)=60\left(-5x+1\right)$ **Solution - YouTube Link**

Example 14. Solve the given equation. Make sure to check your solutions.

$$\left(2x-3\right)^2-\left(x+1\right)\left(3x-5\right)=11-\left(x-1\right)\left(3-x\right)$$

Solution: We carefully expand the indicated products and combine like terms. Notice that even after we expanded $\left(x+1\right)\left(3x-5\right)$ and $\left(x-1\right)\left(3-x\right)$, we still need to keep them in parentheses because we are subtracting them. We will first work out the products.

$\left(2x-3\right)^2=\left(2x-3\right)\left(2x-3\right)=4x^2-6x-6x+9=4x^2-12x+9$

$\left(x+1\right)\left(3x-5\right)=3x^2-5x+3x-5=3x^2-2x-5$

$\left(x-1\right)\left(3-x\right)=3x-x^2-3+x=-x^2+4x-3$

We are now ready to begin to solve the equation.

$$\begin{aligned}\left(2x-3\right)^2-\left(x+1\right)\left(3x-5\right) &= 11-\left(x-1\right)\left(3-x\right) &&\\ 4x^2-12x+9-\left(3x^2-2x-5\right) &= 11-\left(-x^2+4x-3\right) && \text{to subtract is to add the opposite}\\ 4x^2-12x+9-3x^2+2x+5 &= 11+x^2-4x+3 && \text{combine like terms}\\ x^2-10x+14 &= x^2-4x+14 && \text{subtract } x^2\\ -10x+14 &= -4x+14 && \text{add } 10x\\ 14 &= 6x+14 && \text{subtract } 14\\ 0 &= 6x && \text{divide by } 6\\ 0 &= x &&\end{aligned}$$

We check: if $x=0$, then

$\text{LHS}=\left(2\cdot 0-3\right)^2-\left(0+1\right)\left(3\cdot 0-5\right)=\left(-3\right)^2-1\left(-5\right)=9+5=14$

$\text{RHS}=11-\left(0-1\right)\left(3-0\right)=11-\left(-1\right)3=11+3=14$

and so our solution, $\boxed{x=0}$ is correct.

Example 14v. Solve the given equation. $\left(x+2\right)\left(2x-3\right)=2\left(x+1\right)^2-17$ Solution - YouTube Link

Example 15. If we increase the length of each side of a square by $4\,\mathrm{cm}$, the area of the square increases by $64\,\mathrm{cm}^2$. How long are the sides before the increase?

Solution: Let us denote the length of the original square by x. Then its area is x^2. The side of the larger square is $x+4$. Therefore, the area of the larger square is $(x+4)^2$. The equation will express the comparison between the two areas.

$$(x+4)^2 = x^2 + 64 \qquad \text{expand complete square on the left-hand side}$$
$$x^2 + 8x + 16 = x^2 + 64 \qquad \text{subtract } x^2$$
$$8x + 16 = 64 \qquad \text{subtract } 16$$
$$8x = 48 \qquad \text{divide by } 8$$
$$x = 6$$

Thus the original square has sides $6\,\mathrm{cm}$ long. The area is $36\,\mathrm{cm}^2$. If we increased each side by $4\,\mathrm{cm}$, the new side is $10\,\mathrm{cm}$, and the new area $100\,\mathrm{cm}^2$. Indeed, the two areas differ by $100\,\mathrm{cm}^2 - 36\,\mathrm{cm}^2 = 64\,\mathrm{cm}^2$. Thus our solution is correct: the original square has sides that are $\boxed{6\,\mathrm{cm}}$ long.

Example 15v. The area of a square would increase by 17 square-yards if we increased its sides by 1 yard.. How long is a side of the square now (before the increase)? Solution - YouTube Link

Example 16. If we square a number, the result is 29 less than the product of the numbers one greater and two greater than the original number. Find this number.

Solution: Let us denote the number by x. The the numbers one and two greater are $x+1$ and $x+2$. The equation will compare the two products:

$$x^2 = (x+1)(x+2) - 29 \qquad \text{expand products}$$
$$x^2 = x^2 + 2x + x + 2 - 29 \qquad \text{combine like terms}$$
$$x^2 = x^2 + 3x - 27 \qquad \text{subtract } x^2$$
$$0 = 3x - 27 \qquad \text{add } 27$$
$$27 = 3x \qquad \text{divide by } 3$$
$$9 = x$$

Thus this number is $\boxed{9}$. We check: $9^2 = 81$ and $(9+1)(9+2) = 10 \cdot 11 = 110$ and 110 is 29 greater than 81. Thus our solution is correct.

Example 17. Solve each of the given equations. Make sure to check your solutions.

a) $\dfrac{2x-5}{3}-\dfrac{x-2}{5}=x-5$

b) $\dfrac{x+5}{7}-\dfrac{2x+1}{9}=x+6$

Solution - Youtube link

c) $\dfrac{2}{3}(x-1)-\dfrac{1}{2}\left(x+\dfrac{3}{5}\right)=-x+\dfrac{37}{10}$

d) $\dfrac{1}{10}(x-14)=\dfrac{5}{2}x-\dfrac{3}{5}$

Solution - Youtube link

Solution: a) The main idea here is that we can clear denominators of fractions in equations by muliplying by a suitable number. As always, we will multiply both sides.

$$\begin{aligned}
\frac{2x-5}{3}-\frac{x-2}{5} &= x-5 && \text{we write everything as a fraction}\\
\frac{2x-5}{3}-\frac{x-2}{5} &= \frac{x-5}{1} && \text{bring all three fractions to the common denominator}\\
\frac{5(2x-5)}{15}-\frac{3(x-2)}{15} &= \frac{15(x-5)}{15} && \text{clear denominators by multiplying by 15}\\
5(2x-5)-3(x-2) &= 15(x-5) && \text{remove parentheses}\\
10x-25-3x+6 &= 15x-75 && \text{combine like terms}\\
7x-19 &= 15x-75 && \text{subtract } 7x\\
-19 &= 8x-75 && \text{add 75}\\
56 &= 8x && \text{divide by 8}\\
7 &= x
\end{aligned}$$

We check: if $x=7$, then

$$\text{LHS}=\frac{2\cdot 7-5}{3}-\frac{7-2}{5}=\frac{14-5}{3}-\frac{5}{5}=\frac{9}{3}-1=3-1=2 \quad\text{and}\quad \text{RHS}=7-5=2$$

and so our solution, $\boxed{x=7}$ is correct.

b) There are several methods available. The method presented here is focusing how similar this equation is to the previous example.

$$x=1x=\frac{5}{5}x \quad\text{and}\quad -x=-1x=-\frac{10}{10}x$$

$$\begin{aligned}
\frac{2}{3}(x-1)-\frac{1}{2}\left(x+\frac{3}{5}\right) &= -x+\frac{37}{10}\\
\frac{2}{3}(x-1)-\frac{1}{2}\left(\frac{5x}{5}+\frac{3}{5}\right) &= -\frac{10x}{10}+\frac{37}{10}\\
\frac{2}{3}\cdot\frac{x-1}{1}-\frac{1}{2}\cdot\frac{5x+3}{5} &= \frac{-10x+37}{10}\\
\frac{2(x-1)}{3}-\frac{5x+3}{10} &= \frac{-10x+37}{10} && \text{bring all fractions to the common denominator}\\
\frac{20(x-1)}{30}-\frac{3(5x+3)}{30} &= \frac{3(-10x+37)}{30} && \text{clear denominator by multiplying by 30}\\
20(x-1)-3(5x+3) &= 3(-10x+37) && \text{remove parentheses}\\
20x-20-15x-9 &= -30x+111 && \text{combine like terms}\\
5x-29 &= -30x+111 && \text{add } 30x\\
35x-29 &= 111 && \text{add 29}\\
35x &= 140 && \text{divide by 35}\\
x &= 4
\end{aligned}$$

We check: if $x=4$, then

$$\begin{aligned}\text{LHS} &= \frac{2}{3}(4-1)-\frac{1}{2}\left(4+\frac{3}{5}\right)=\frac{2}{3}\cdot 3-\frac{1}{2}\left(\frac{20}{5}+\frac{3}{5}\right)=2-\frac{1}{2}\cdot\frac{23}{5}=2-\frac{23}{10}=\frac{20}{10}-\frac{23}{10}=\frac{-3}{10}\\ \text{RHS} &= -4+\frac{37}{10}=-\frac{40}{10}+\frac{37}{10}=-\frac{3}{10}\end{aligned}$$

and so our solution, $\boxed{x=4}$ is correct.

Part 4 - Formulas

Sometimes we will solve equations in more abstract forms. The following examples are called **formulas** or **literal equations**. While they might be intimidating for students, the ideas and techniques are the same.

Example 18. Solve each of the given equations for the specified variable.

a) Solve $A+B=C$ for A b) $IR=V$ for I

Solution: a) All of A, B, and C are unknown, but the instructions identify A as the unknown in which we are interested. We preted that we know the values of B and C, we just don't care about them. So, first we ask: *what happened to our unknown*? On the left-hand side, we see $A+B$. This means that someone came along and added B to the unknown A. In order to isolate the unknown A, we need to 'undo' this operation, that is, we will subtract B from both sides. Although we do not know the value of B, we still are subtracting the same amount when subtracting B.

$$A+B=C \qquad \text{subtract } B$$
$$A=C-B$$

What is somewhat unsettling is that the expression $C-B$ does not collapse to a number because their values are not known. Either way, the solution is $\boxed{A=C-B}$. We can still check, if $A=C-B$, then the left-hand side is

$$\text{LHS}=A+B=\underbrace{C-B}_{A}+B=C-B+B=C=\text{RHS}\ \checkmark$$

So our solution, $\boxed{A=C-B}$ is correct.

b) Consider now the equation $IR=V$. This equation is from physics, it is called Ohm's law. If we connect a lightbulb to a battery, the electric current created depends on properties of the light bulb and the battery. Ohm's law expresses the connection between resistance of the lightbulb (denoted by R), the potential or voltage of the battery (denoted by V), and the electric current (denoted by I). Our unknown is I. The unknown was multiplied by R. In order to isolate the unknown, we will divide both sides by R.

$$IR=V \qquad \text{divide by } R$$
$$I=\frac{V}{R}$$

So the only solution of this equation is $\boxed{I=\dfrac{V}{R}}$.

If we can compute with formulas in the abstract, one formula becomes many. We can solve $V=IR$ for I and get $I=\dfrac{V}{R}$ and also, solve $V=IR$ for R and get $R=\dfrac{V}{I}$.

There is an application of one-step equations that helps us with a tricky algebraic expression that comes up often. Suppose we want to express that two numbers add up to 10. If label one number by x, how can we label the other number?

Example 19. Suppose that x represents a number. Let y be another number such that the sum of x and y is 10. Express y in terms of x.

Solution: Our numbers are labeled x and y. We state that their sum is 10 and solve the equation for y in terms of x.

$$x + y = 10 \qquad \text{subtract } x$$
$$y = 10 - x$$

So y can be expressed in terms of x as $\boxed{10 - x}$.

Caution! $x - 10$ and $10 - x$ look similar but they are very different. $x - 10$ is a number ten less than x, while $10 - x$ is the number, that, when added to x, results in 10. If we confuse the two, we can quickly check which is which by evaluating the expressions using a few numbers for x. We come up with a few values for x, say -10, -5, 1, 6, 10, and 20. Then we evaluate $x - 10$ and $10 - x$ using these values for x.

x	-10	-5	1	6	10	20
$x - 10$	-20	-15	-9	-4	0	10

and

x	-10	-5	1	6	10	20
$10 - x$	20	15	9	4	0	-10

We can now easily tell which table's columns add up to 10 and which table has columns in which the second number is ten less than the first one. We needed a few values because sometimes we can get unlucky: notice that if x is 10, then both $10 - x$ and $x - 10$ give us the same zero.

Example 20. Solve each of the given equations for the unknown indicated.

a) $A = 3B - C$ for B b) $A = 3(B - C)$ for B

Solution: a) We are to solve the equation $A = 3B - C$ for B. Two things happened to the unknown: first a multiplication by 3 and then C was subtracted. To isolate B, we will reverse those operations in the reverse order. This means that we will first add C and then divide by 3.

$$A = 3B - C \qquad \text{add } C$$
$$A + C = 3B \qquad \text{divide by 3}$$
$$\frac{A + C}{3} = B \qquad \text{and so } \boxed{B = \frac{A + C}{3}}.$$

b) The equation $A = 3(B - C)$ is very similar to the previous one, because it involves the same two operations; only the order is different. We first subtract C and then multiply by 3. So we will divide by 3 first and then add C.

$$\begin{aligned} A &= 3(B - C) && \text{divide by 3} \\ \frac{A}{3} &= B - C && \text{add } C \\ \frac{A}{3} + C &= B \end{aligned}$$

Thus our solution is $\boxed{B = \frac{A}{3} + C}$.

Often a presented method is not the only one possible. We can solve this equation differently, by first distributing 3 and then basically solving an equation very similar to the previous example.

$$\begin{aligned} A &= 3(B-C) && \text{distribute 3} \\ A &= 3B-3C && \text{add } 3C \\ A+3C &= 3B && \text{divide by 3} \\ \frac{A+3C}{3} &= B \end{aligned}$$

Both methods are correct, and the two results are the same, although they might appear different at first. Once we have more algebra skills under our belt, we will be able to verify that the two expressions are really the same.

Example 21. Temperature can be measured in Farenheit (F) and in Celsius (C). Given the temperature in celsius, it can be converted to Farenheit using the formula $F = \frac{9}{5}C + 32$

a) Convert 35 celsius to farenheit.

b) Convert 122 farenheit to celsius.

c) Solve the formula given above for C.

d) Is it possible for the same number to refer to a temperature, whether it is in celsius or in farenheit?

Solution: a) We substitute $C = 35$ into the formula given.

$$F = \frac{9}{5}C + 32 = F = \frac{9}{5} \cdot 35 + 32 = 63 + 32 = 95$$

So 35 celsius is the same as $\boxed{\text{95 farenheit.}}$

b) We substitute the given data into the same formula. Then we solve the ensuing linear equation for C.

$F = \frac{9}{5}C + 32$ becomes $122 = \frac{9}{5}C + 32$ Let's solve for C.

$$\begin{aligned} 122 &= \frac{9}{5}C + 32 && \text{subtract 32} \\ 90 &= \frac{9}{5}C && \text{divide by } \frac{9}{5} \\ C &= \frac{90}{\frac{9}{5}} = 90 \cdot \frac{5}{9} = 50 \end{aligned}$$

So 122 farenheit is 50 celsius degrees.

c) We will perform the same steps as before but use C in the abstract instead of using a specific number. We solve for C.

$$\begin{aligned} F &= \frac{9}{5}C + 32 && \text{subtract 32} \\ F - 32 &= \frac{9}{5}C && \text{divide by } \frac{9}{5} \\ C &= \frac{F-32}{\frac{9}{5}} = (F-32)\,\frac{5}{9} \end{aligned}$$

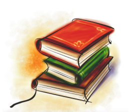
So the answer is either $\boxed{C = \frac{5}{9}(F-32)}$ or, if we distribute, $C = \frac{5}{9}F - \frac{160}{9}$.

d) Let x be, if exists, a temperature in celsius given. if we convert this to farenheit, we get that it is x. So from

$$F = \frac{9}{5}C + 32 \text{ we write } \quad x = \frac{9}{5}x + 32 \quad \text{and solve for } x$$

$$\begin{aligned} x &= \frac{9}{5}x + 32 && \text{subtract } x \qquad \frac{9}{5} - 1 = \frac{4}{5} \\ 0 &= \frac{4}{5}x + 32 && \text{subtract } 32 \\ -32 &= \frac{4}{5}x && \text{divide by } \frac{4}{5} \\ x &= \frac{-32}{\frac{4}{5}} = -32 \cdot \frac{5}{4} = -40 \end{aligned}$$

Indeed, $\boxed{-40}$ celsius and -40 farenheit are the same temperature.



Sample Problems

Solve each of the following equations. Make sure to check your solutions.

1. $2x + 3 = 4x + 9$
2. $3w - 5 = 5(w + 1)$
3. $3y - 9 = -2y + 4$
4. $4 - x = 3(x - 7)$
5. $7(j - 5) + 9 = 2(-2j + 5) + 5j$
6. $3(x - 5) - 5(x - 1) = -2x + 1$
7. $\dfrac{3 - x}{4} - \dfrac{10 - 3x}{5} = x + 2$
8. $\dfrac{3x + 17}{2} = x - 1 + \dfrac{x + 19}{2}$
9. $\dfrac{2}{3}(x - 1) = \dfrac{3}{5}(x - 4) + 1$
10. $\dfrac{2}{3}(x - 7) = \dfrac{4}{5}(x + 1)$
11. $\dfrac{x + 2}{4} - \dfrac{x - 3}{5} = 20 - x$
12. $(x - 3)^2 - (2x - 5)(x + 1) = 5 - (x - 1)^2$
13. $(x + 1)^2 - (2x - 1)^2 + (3x)^2 = 6x(x - 2)$
14. $12 - (2p - 1)(p + 1) = -2(-p + 5)^2$
15. $\dfrac{\dfrac{\frac{5x - 1}{7} + 3}{5} - 10}{3} = -4$
16. Paul invested his money on the stock market. First he bet on a risky stock and lost half of his money. Then he became a bit more careful and invested money in more conservative stocks that involved less risk but also less profit. His investments made him 80 dollars. If he has 250 dollars in the stock market today, with how much money did he start investing?
17. In a hotel, the first night costs 45 dollars, and all additional nights cost 35 dollars. How long did Mr. Williams stay in the hotel if his bill was 325 dollars?
18. If we increase each side of a square by $3\,\mathrm{cm}$, its area increases by $51\,\mathrm{cm}^2$. How long are the sides before the increase?

19. If we add two to a number and multiply that by one less than that number, the product is 6 less than the square of the number. Find this number.

20. One side of a rectangle is three less than twice another side. If we increase both sides by 1 unit, the area of the rectangle will increase by 10 units. Find the sides of the original rectangle.

21. Solve each of the following formulas.

a) $3x - 4y = z$ for x b) $3x - 4y = z$ for y c) $AB + PQ = S$ for Q

Practice Problems

Solve each of the following equations. Make sure to check your solutions.

1. $5x - 3 = x + 9$
2. $-x + 13 = 2x + 1$
3. $-2x + 4 = 5x - 10$
4. $5x - 7 = 6x + 8$
5. $8x - 1 = 3x + 19$
6. $-7x - 1 = 3x - 21$
7. $3(x-4) = 2(x+5)$
8. $4(5x+1) = 6x + 4$
9. $a - 3 = 5(a-1) - 2$
10. $3y - 2 = -2y + 18$
11. $8(x-3) - 3(5-2x) = x$
12. $5(x-1) - 3(x+1) = 3x - 8$
13. $-2x - (3x-1) = 2(5-3x)$
14. $3(x-4) + 5(x+8) = 2(x-1)$
15. $5(x-1) - 3(-x+1) = -3 + 8x$
16. $\dfrac{3x-1}{5} - \dfrac{7-x}{3} = 2x + 6$
17. $\dfrac{3x-1}{4} + \dfrac{8-4x}{3} = -3 - x$
18. $\dfrac{3x-2}{5} + \dfrac{x+4}{3} = \dfrac{14(x+1)}{15}$
19. $\dfrac{3}{8}x + 1\dfrac{4}{5} = \dfrac{1}{4}x + 1\dfrac{3}{10}$
20. $\dfrac{2x+1}{3} - \dfrac{3-x}{2} = x - 2$
21. $\dfrac{2}{3}x - 1 = -\dfrac{2}{3}\left(x + \dfrac{1}{2}\right)$
22. $2(b+1) - 5(b-3) = 2(b-7) + 1$
23. $3(2x-1) - 5(2-x) = 4(x-1) + 5$
24. $3(2x-7) - 2(5x+2) = -5x - 30$
25. $3(x-4) - 4(x-3) = 3(x-2) + 2(3-x)$
26. $2x(3x-1) - x(5x-2) = (x-1)^2$
27. $y^2 - (y-1)^2 + (y-2)^2 = (y-3)(y-5)$
28. $(3x)^2 - (x+3)(5x-3) = (5-2x)^2 - 16$
29. $(w+4)(1-2w) = 3w - 2(w-3)^2$
30. $(2x-3)^2 - 3(x-2)^2 = 10 - (x-2)(7-x)$
31. $(2-w)^2 - (2w-3)^2 + 7 = (w-2)(5-3w)$
32. $3(a+11) - a(8-3a) = 3(a-2)^2$
33. $-5(2x-1) - (4-x)^2 = 3 - (x+1)^2$
34. $5(-3-x) - 3x(x-2) = x - 3(x+2)(x-5)$
35. $2(-m-2)^2 - (m-2)^2 = 8m + (m+2)^2$
36. $(3a-5)(2-a) - (2a-1)(a+3) = -5a^2 - 7$
37. $\dfrac{1}{2}(x-3)^2 - \dfrac{1}{2}(x+1)^2 = 4(x-7)$
38. $\dfrac{\dfrac{\dfrac{3x-1}{2}+4}{-5} - 6}{2} + 7 = 2$
39. $\dfrac{\dfrac{5\left(\dfrac{5x+2}{-7} - 7\right) + 11}{2} + 12}{5} = 2$

40. Three times the difference of x and 7 is -15. Find x.

41. Ann and Bonnie are discussing their financial situation. Ann said: *If you take* 50 *bucks from me and then doubled what is left, I would have* \$300. Bonnie answers: *That's funny. If you doubled my money first and then took* \$50, *then I would have* \$300! How much do they each have?

42. If we increase the length of each side of a square by $2\,\mathrm{cm}$, the area of the square increases by $24\,\mathrm{cm}^2$. How long are the sides before the increase?

43. If we square the sum of a number and three, the result is twelve less than the square of the sum of the number and one. Find this number.

44. The product of two consecutive numbers is 8 less than the square of the greater number. Find these numbers.

45. One side of a rectangle is 16 inches shorter than five times another side. How long are the sides if the perimeter of the rectangle is 136 inches.

46. A pound of cashew costs 5 dollars, a pound of peanut costs 3 dollars. The peanut purchased was one pound les than twice the weight of the cashew purchased. How many pounds of each did we buy if we paid a total of 85 dollars?

47. Solve each of the formulas.
a) $\dfrac{a+b}{2}=c$ for a b) $5x+3y=15$ for y c) $M(F-S)=C$ for F

Answers

Discussion

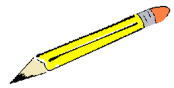
a) contradiction b) conditional c) conditional d) identity

Sample Problems

1. -3 2. -5 3. $\dfrac{13}{5}$ 4. $\dfrac{25}{4}$ 5. 6 6. no solution 7. -5 8. identity, all numbers are solution
9. -11 10. -41 11. 18 12. 2 13. 0 14. 3 15. -18 16. \$340 17. 9 18. $7\,\mathrm{cm}$
19. -4 20. 4 units by 5 units 21. a) $x=\dfrac{z+4y}{3}$ b) $y=\dfrac{3x-z}{4}$ c) $Q=\dfrac{S-AB}{P}$

Practice Problems

1. 3 2. 4 3. 2 4. -15 5. 4 6. 2 7. 22 8. 0 9. 1 10. 4 11. 3 12. 0 13. 9 14. -5
15. there is no solution 16. -8 17. -13 18. all real numbers are solution 19. -4 20. -5 21. $\dfrac{1}{2}$
22. 6 23. 2 24. -5 25. 0 26. $\dfrac{1}{2}$ 27. 2 28. 0 29. 1 30. 3 31. 4 32. -3
33. there is no solution 34. -5 35. all real numbers are solution 36. 0 37. 4 38. 11 39. -6 40. 2
41. Ann has 200 dollars, Bonnie has 175 dollars 42. $5\,\mathrm{cm}$ 43. -5 44. 7 45. 14 in by 54 in
46. 8 lbs cashews and 15 lbs of peanuts 47. a) $a=2c-b$ b) $y=-\dfrac{5}{3}x+5$ for y c) $\dfrac{C+MS}{M}$

Sample Problems  Solutions

1. $2x+3=4x+9$

Solution:

$$\begin{aligned} 2x+3 &= 4x+9 && \text{subtract } 2x \text{ from both sides} \\ 3 &= 2x+9 && \text{subtract } 9 \text{ from both sides} \\ -6 &= 2x && \text{divide both sides by 2} \\ -3 &= x \end{aligned}$$

We check: if $x=-3$, then

$$\begin{aligned} \text{LHS} &= 2(-3)+3=-6+3=-3 \\ \text{RHS} &= 4(-3)+9=-12+9=-3 \end{aligned}$$

Thus our solution, $\boxed{x=-3}$ is correct. (Note: LHS is short for the left-hand side and RHS is short for the right-hand side.)

2. $3w-5=5(w+1)$

Solution: we first apply the law of distributivity to simplify the right-hand side.

$$\begin{aligned} 3w-5 &= 5(w+1) \\ 3w-5 &= 5w+5 && \text{subtract } 3w \text{ from both sides} \\ -5 &= 2w+5 && \text{subtract } 5 \text{ from both sides} \\ -10 &= 2w && \text{divide both sides by 2} \\ -5 &= w \end{aligned}$$

We check. If $w=-5$, then

$$\begin{aligned} \text{LHS} &= 3(-5)-5=-15-5=-20 \\ \text{RHS} &= 5((-5)+1)=5(-4)=-20 \end{aligned}$$

Thus our solution, $\boxed{w=-5}$ is correct.

3. $3y-9=-2y+4$

Solution:

$$\begin{aligned} 3y-9 &= -2y+4 && \text{add } 2y \text{ to both sides} \\ 5y-9 &= 4 && \text{add } 9 \text{ to both sides} \\ 5y &= 13 && \text{divide both sides by } 5 \\ y &= \frac{13}{5} \end{aligned}$$

We check. If $y=\frac{13}{5}$, then

$$\begin{aligned} \text{LHS} &= 3\left(\frac{13}{5}\right)-9=\frac{3}{1}\cdot\frac{13}{5}-9=\frac{39}{5}-\frac{9}{1}=\frac{39}{5}-\frac{45}{5}=\frac{-6}{5}=-\frac{6}{5} \\ \text{RHS} &= -2\left(\frac{13}{5}\right)+4=\frac{-2}{1}\cdot\frac{13}{5}+\frac{4}{1}=\frac{-26}{5}+\frac{20}{5}=\frac{-6}{5}=-\frac{6}{5} \end{aligned}$$

Thus $\boxed{y=\frac{13}{5}}$ is the correct solution.

4. $4-x=3(x-7)$
Solution: We first apply the law of distributivity to simplify the right-hand side.

$$\begin{aligned} 4-x &= 3(x-7) && \text{distribute } 3 \\ 4-x &= 3x-21 && \text{add } x \text{ to both sides} \\ 4 &= 4x-21 && \text{add } 21 \text{ to both sides} \\ 25 &= 4x && \text{divide both sides by } 4 \\ \frac{25}{4} &= x \end{aligned}$$

We check. If $x=\frac{25}{4}$, then

$$\begin{aligned} \text{LHS} &= 4-x=4-\frac{25}{4}=\frac{4}{1}-\frac{25}{4}=\frac{16}{4}-\frac{25}{4}=\frac{16-25}{4}=\frac{-9}{4}=-\frac{9}{4} \\ \text{RHS} &= 3(x-7)=3\left(\frac{25}{4}-7\right)=3\left(\frac{25}{4}-\frac{7}{1}\right)=3\left(\frac{25}{4}-\frac{28}{4}\right)=3\left(\frac{25-28}{4}\right) \\ &= 3\left(\frac{-3}{4}\right)=\frac{3}{1}\cdot\frac{-3}{4}=\frac{-9}{4}=-\frac{9}{4} \end{aligned}$$

Thus our solution, $\boxed{x=\frac{25}{4}}$ is correct.

5. $7(j-5)+9=2(-2j+5)+5j$
Solution:

$$\begin{aligned} 7(j-5)+9 &= 2(-2j+5)+5j && \text{distribute on both sides} \\ 7j-35+9 &= -4j+10+5j && \text{combine like terms} \\ 7j-26 &= j+10 && \text{subtract } j \\ 6j-26 &= 10 && \text{add } 26 \\ 6j &= 36 && \text{divide by } 6 \\ j &= 6 \end{aligned}$$

We check: if $j=6$, then

$$\begin{aligned} \text{LHS} &= 7(6-5)+9=7\cdot 1+9=7+9=16 \\ \text{RHS} &= 2(-2\cdot 6+5)+5\cdot 6=2\,(-12+5)+30=2\,(-7)+30=-14+30=16 \end{aligned}$$

Thus our solution, $\boxed{j=6}$ is correct.

6. $3\,(x-5)-5\,(x-1)=-2x+1$
Solution:

$$\begin{aligned} 3\,(x-5)-5\,(x-1) &= -2x+1 && \text{multiply out parentheses} \\ 3x-15-5x+5 &= -2x+1 && \text{combine like terms} \\ -2x-10 &= -2x+1 && \text{add } 2x \\ -10 &= 1 \end{aligned}$$

Since x disappeared from the equation and we are left with an unconditionally false statement, $\boxed{\text{there is no solution}}$ for this equation. This type of an equation is called a **contradiction**.

7. $\dfrac{3-x}{4}-\dfrac{10-3x}{5}=x+2$
Solution:

$$\begin{aligned}
\frac{3-x}{4}-\frac{10-3x}{5} &= x+2 && \text{make everything a fraction}\\
\frac{3-x}{4}-\frac{10-3x}{5} &= \frac{x+2}{1} && \text{common denominator}\\
\frac{5(3-x)}{20}-\frac{4(10-3x)}{20} &= \frac{20(x+2)}{20} && \text{multiply by 20}\\
5(3-x)-4(10-3x) &= 20(x+2) && \text{distribute}\\
15-5x-40+12x &= 20x+40 && \text{combine like terms}\\
7x-25 &= 20x+40 && \text{subtract } 7x\\
-25 &= 13x+40 && \text{subtract } 40\\
-65 &= 13x && \text{divide by 13}\\
-5 &= x
\end{aligned}$$

We check:

$$\begin{aligned}
\text{LHS} &= \frac{3-(-5)}{4}-\frac{10-3(-5)}{5}=\frac{8}{4}-\frac{25}{5}=2-5=-3\\
\text{RHS} &= -5+2=-3
\end{aligned}$$

Thus our solution, $\boxed{x=-5}$ is correct.

8. $\dfrac{3x+17}{2}=x-1+\dfrac{x+19}{2}$
Solution:

$$\begin{aligned}
\frac{3x+17}{2} &= x-1+\frac{x+19}{2} && \text{express everything as a fraction}\\
\frac{3x+17}{2} &= \frac{x-1}{1}+\frac{x+19}{2} && \text{bring everything to the common denominator}\\
\frac{3x+17}{2} &= \frac{2(x-1)}{2}+\frac{x+19}{2} && \text{add fractions on right hand side}
\end{aligned}$$

$$\begin{aligned}
\frac{3x+17}{2} &= \frac{2(x-1)+x+19}{2} && \text{multiply out parentheses}\\
\frac{3x+17}{2} &= \frac{2x-2+x+19}{2} && \text{combine like terms}\\
\frac{3x+17}{2} &= \frac{3x+17}{2} && \text{multiply by 2}\\
3x+17 &= 3x+17
\end{aligned}$$

Because the left hand side is now identical to the right hand side, this equation is an identity, and $\boxed{\text{all real numbers}}$ are solution.

9. $\frac{2}{3}(x-1)=\frac{3}{5}(x-4)+1$
Solution: We re-write the expressions as fractions.

$$\begin{aligned}
\frac{2(x-1)}{3} &= \frac{3(x-4)}{5}+\frac{1}{1} && \text{common denominator is } 15\\
\frac{5\cdot 2(x-1)}{5\cdot 3} &= \frac{3\cdot 3(x-4)}{3\cdot 5}+\frac{15}{15} \\
\frac{10(x-1)}{15} &= \frac{9(x-4)}{15}+\frac{15}{15} && \text{multiply by } 15\\
10(x-1) &= 9(x-4)+15 && \text{distribute}\\
10x-10 &= 9x-36+15 && \text{combine like terms}\\
10x-10 &= 9x-21 && \text{subtract } 9x\\
x-10 &= -21 && \text{add } 10\\
x &= -11
\end{aligned}$$

We check. If $x=-11$, then

$$\begin{aligned}
\text{LHS} &= \frac{2}{3}(-11-1)=\frac{2}{3}(-12)=-8\\
\text{RHS} &= \frac{3}{5}(-11-4)+1=\frac{3}{5}(-15)+1=-9+1=-8
\end{aligned}$$

Thus our solution, $\boxed{x=-11}$ is correct.

10. $\frac{2}{3}(x-7)=\frac{4}{5}(x+1)$
Solution:

$$\begin{aligned}
\frac{2}{3}(x-7) &= \frac{4}{5}(x+1)\\
\frac{2}{3}\cdot\frac{x-7}{1} &= \frac{4}{5}\cdot\frac{x+1}{1}\\
\frac{2(x-7)}{3} &= \frac{4(x+1)}{5} && \text{bring fractions to common denominator}\\
\frac{5\cdot 2(x-7)}{15} &= \frac{3\cdot 4(x+1)}{15} && \text{multiply both sides by } 15
\end{aligned}$$

$$\begin{aligned}
10(x-7) &= 12(x+1) && \text{multiplty out parentheses}\\
10x-70 &= 12x+12 && \text{subtract } 10x\\
-70 &= 2x+12 && \text{subtract } 12\\
-82 &= 2x && \text{divide by } 2\\
-41 &= x
\end{aligned}$$

We check:

$$\begin{aligned}
\text{LHS} &= \frac{2}{3}(-41-7)=\frac{2}{3}(-48)=-32\\
\text{RHS} &= \frac{4}{5}(-41+1)=\frac{4}{5}(-40)=-32
\end{aligned}$$

Thus our solution, $\boxed{x=-41}$ is correct.

11. $\dfrac{x+2}{4}-\dfrac{x-3}{5}=20-x$

Solution:

$$\begin{aligned}
\frac{x+2}{4}-\frac{x-3}{5} &= 20-x && \text{make everything a fraction}\\
\frac{x+2}{4}-\frac{x-3}{5} &= \frac{20-x}{1} && \text{common denominator is 20}\\
\frac{5(x+2)}{20}-\frac{4(x-3)}{20} &= \frac{20(20-x)}{20} && \text{multiply by 20}\\
5(x+2)-4(x-3) &= 20(20-x) && \text{distribute}\\
5x+10-4x+12 &= 400-20x && \text{combine like terms}\\
x+22 &= -20x+400 && \text{add } 20x\\
21x+22 &= 400 && \text{subtract 22}\\
21x &= 378 && \text{divide by 21}\\
x &= 18
\end{aligned}$$

We check. If $x=18$, then

$$\begin{aligned}
\text{LHS} &= \frac{18+2}{4}-\frac{18-3}{5}=\frac{20}{4}-\frac{15}{5}=5-3=2\\
\text{RHS} &= 20-18=2
\end{aligned}$$

Thus $\boxed{x=18}$ is indeed the solution.

12. $(x-3)^2-(2x-5)(x+1)=5-(x-1)^2$

Solution: We first multiply the polynomials as indicated. If the product is subtracted or further multiplied, we must keep the parentheses.

$$\begin{aligned}
(x-3)^2-(2x-5)(x+1) &= 5-(x-1)^2\\
x^2-3x-3x+9-\left(2x^2+2x-5x-5\right) &= 5-\left(x^2-x-x+1\right) && \text{combine like terms}\\
x^2-6x+9-\left(2x^2-3x-5\right) &= 5-\left(x^2-2x+1\right) && \text{distribute}\\
x^2-6x+9-2x^2+3x+5 &= 5-x^2+2x-1 && \text{combine like terms}\\
-x^2-3x+14 &= -x^2+2x+4 && \text{add } x^2\\
-3x+14 &= 2x+4 && \text{add } 3x\\
14 &= 5x+4 && \text{subtract 4}\\
10 &= 5x && \text{divide by 5}\\
2 &= x
\end{aligned}$$

We check. If $x=2$, then

$$\begin{aligned}
\text{LHS} &= (2-3)^2-(2\cdot 2-5)(2+1)=(-1)^2-(4-5)(2+1)=(-1)^2-(-1)\cdot 3\\
&= 1-(-3)=4\\
\text{RHS} &= 5-(2-1)^2=5-1^2=5-1=4
\end{aligned}$$

Thus $\boxed{x=2}$ is indeed the solution.

13. $(x+1)^2-(2x-1)^2+(3x)^2=6x(x-2)$

Solution: We first multiply the polynomials as indicated. If the product is subtracted or further multiplied, we must keep the parentheses.

$$\begin{aligned}
(x+1)^2-(2x-1)^2+(3x)^2 &= 6x(x-2) \\
x^2+x+x+1-\left(4x^2-2x-2x+1\right)+9x^2 &= 6x^2-12x \\
x^2+2x+1-\left(4x^2-4x+1\right)+9x^2 &= 6x^2-12x && \text{distribute} \\
x^2+2x+1-4x^2+4x-1+9x^2 &= 6x^2-12x && \text{combine like terms} \\
6x^2+6x &= 6x^2-12x && \text{subtract } 6x^2 \\
6x &= -12x && \text{add } 12x \\
18x &= 0 && \text{divide by } 18 \\
x &= 0
\end{aligned}$$

We check. If $x=0$, then

$$\begin{aligned}
\text{LHS} &= (0+1)^2-(2\cdot 0-1)^2+(3\cdot 0)^2=1^2-(-1)^2+(0)^2 \\
&= 1-1+0=0 \\
\text{RHS} &= 6\cdot 0\cdot(0-2)=6\cdot 0\cdot(-2)=0
\end{aligned}$$

Thus $\boxed{x=0}$ is indeed the solution.

14. $12-(2p-1)(p+1)=-2(-p+5)^2$

Solution: We first multiply the polynomials as indicated. If the product is subtracted or further multiplied, we must keep the parentheses.

$$\begin{aligned}
12-(2p-1)(p+1) &= -2(-p+5)^2 \\
12-\left(2p^2+2p-p-1\right) &= -2\left(p^2-5p-5p+25\right) && \text{combine like terms} \\
12-\left(2p^2+p-1\right) &= -2\left(p^2-10p+25\right) && \text{distribute} \\
12-2p^2-p+1 &= -2p^2+20p-50 && \text{combine like terms} \\
-2p^2-p+13 &= -2p^2+20p-50 && \text{add } 2p^2 \\
-p+13 &= 20p-50 && \text{add } p \\
13 &= 21p-50 && \text{add } 50 \\
63 &= 21p && \text{divide by } 21 \\
3 &= p
\end{aligned}$$

We check. If $p=3$, then

$$\begin{aligned}
\text{LHS} &= 12-(2\cdot 3-1)(3+1)=12-(6-1)(3+1)=12-5\cdot 4=12-20=-8 \\
\text{RHS} &= -2(-3+5)^2=-2\cdot 2^2=-2\cdot 4=-8
\end{aligned}$$

Thus $\boxed{p=3}$ is indeed the solution.

15. $\dfrac{\dfrac{\dfrac{5x-1}{7}+3}{5}-10}{3}=-4$

Soluton: This is one of those many-step equations where we have more than two steps, but we simply perform them in the reverse order like we did in the case of two-step equations.

$$\begin{aligned}
\dfrac{\dfrac{\dfrac{5x-1}{7}+3}{5}-10}{3} &= -4 && \text{multiply by 3}\\
\dfrac{\dfrac{5x-1}{7}+3}{5}-10 &= -12 && \text{add 10}\\
\dfrac{\dfrac{5x-1}{7}+3}{5} &= -2 && \text{multiply by 5}\\
\dfrac{5x-1}{7}+3 &= -10 && \text{subtract 3}\\
\dfrac{5x-1}{7} &= -13 && \text{multiply by 7}\\
5x-1 &= -91 && \text{add 1}\\
5x &= -90 && \text{divide by 5}\\
x &= -18
\end{aligned}$$

We check: if $x=-18$, then

$$\begin{aligned}
\text{LHS} &= \dfrac{\dfrac{\dfrac{5(-18)-1}{7}+3}{5}-10}{3} = \dfrac{\dfrac{\dfrac{-90-1}{7}+3}{5}-10}{3} = \dfrac{\dfrac{\dfrac{-91}{7}+3}{5}-10}{3} = \dfrac{\dfrac{-13+3}{5}-10}{3}\\
&= \dfrac{\dfrac{-10}{5}-10}{3} = \dfrac{-2-10}{3} = \dfrac{-12}{3} = -4 = \text{RHS} \checkmark
\end{aligned}$$

Thus our solution, $\boxed{x=-18}$ is correct.

16. Paul invested his money on the stock market. First he bet on a risky stock and lost half of his money. Then he became a bit more careful and invested money in more conservative stocks that involved less risk but also less profit. His investments made him 80 dollars. If he has 250 dollars in the stock market today, with how much money did he start investing?

Solution: Let us denote the amount of money with which Paul started to invest by x. First he lost half of his money, so he had $\dfrac{x}{2}$. Then he gained 80 dollars end ended up with 250 dollars. So, we can write the equation $\dfrac{x}{2}+80=250$. We will solve this two-step equation for x. What happened to the unknown was first division by 2 and then addition of 80. To reverse these operations, we will first subtract 80 and then multiply by 2.

$$\begin{aligned}
\dfrac{x}{2}+80 &= 250 && \text{subtract 80}\\
\dfrac{x}{2} &= 170 && \text{multiply by 2}\\
x &= 340
\end{aligned}$$

So Paul started with 340 dollars. We check: If we lose half of 340,dollars we have 170 dollars left. Then when we add 80,dollars we indeed end up with 250 dollars. So our solution is correct,
$\boxed{\text{Paul started with 340 dollars}}$.

17. In a hotel, the first night costs 45 dollars, and all additional nights cost 35 dollars. How long did Mr. Williams stay in the hotel if his bill was 325 dollars?

Solution: Suppose that Mr. Williams stayed for the first night and the an additional x many nights. Then the bill would be $45 + x \cdot 35$ or $35x + 45$. So we write and then solve the equation $35x + 45 = 325$.

$$\begin{aligned} 35x + 45 &= 325 && \text{subtract } 45 \\ 35x &= 280 && \text{divide by } 35 \\ x &= 8 \end{aligned}$$

Thus Mr. Williams stayed in the hotel for 9 nights. Why not 8 if we got $x = 8$? Remember, the first night was counted separately; there was the first night and then $x = 8$ additional nights. This is why it is a good idea to read the text of the problem one more time before we state our final answer. So $\boxed{\text{Mr. Williams stayed 9 nights in the hotel}}$. We check: the bill for 9 nights would be $45 + 8(35) = 325$, and so our solution is correct.

18. If we increase each side of a square by $3\,\text{cm}$, its area increases by $51\,\text{cm}^2$. How long are the sides before the increase?

Solution: Let us denote the length of the sides of the original square by x. Then the area of the square is x^2. The side of the larger square is $x + 3$. Therefore, the area of the larger square is $(x + 3)^2$. The equation will express the comparison between the two areas.expand complete square on the left-hand side

$$\begin{aligned} (x + 3)^2 &= x^2 + 51 && \text{expand complete square on the left-hand side} \\ x^2 + 6x + 9 &= x^2 + 51 && \text{subtract } x^2 \\ 6x + 9 &= 51 && \text{subtract } 9 \\ 6x &= 42 && \text{divide by } 6 \\ x &= 7 \end{aligned}$$

Thus the original square has sides $7\,\text{cm}$ long. The area is $49\,\text{cm}^2$. If we increased each side by $3\,\text{cm}$, the new side is $10\,\text{cm}$, and the new area $100\,\text{cm}^2$. Indeed, the two areas differ by $100\,\text{cm}^2 - 49\,\text{cm}^2 = 51\,\text{cm}^2$. Thus our solution is correct: the original square has sides that are $\boxed{7\,\text{cm}}$ long.

19. If we add two to a number and multiply that by one less than that number, the product is 6 less than the square of the number. Find this number.

Solution: Let x denote this number. Then one quantity compared is $(x + 2)(x - 1)$, the other quantity compared is x^2, and we express the comparison when setting up the equation.

$$(x + 2)(x - 1) = x^2 - 6$$

We then solve for x. We expand the product as margin-work. $(x + 2)(x - 1) = x^2 - x + 2x - 2$
$= x^2 + x - 2$

$$\begin{aligned} x^2 + x - 2 &= x^2 - 6 && \text{subtract } x^2 \\ x - 2 &= -6 && \text{add } 2 \\ x &= -4 \end{aligned}$$

So our number is -4. We check against the conditions stated in the problem. The product of two more and one less of our number is then $-2(-5) = 10$. This number is indeed 6 less than 16, which is the square of our number, $\overline{-4}$. So our solution is correct.

20. One side of a rectangle is three less than twice another side. If we increase both sides by 1 unit, the area of the rectangle will increase by 10 square-units. Find the sides of the original rectangle.

Solution: Let us denote one side of the original rectangle by x. Then the other side in the rectangle is $2x-3$. The area of this rectangle is then $x(2x-3)$.

If we increase both sides, then the sides become $x+1$ and $2x-2$. This rectangle has an area $(x+1)(2x-2)$. The equation will express a comparison between the two areas.

$$\begin{aligned}(x+1)(2x-2) &= x(2x-3)+10 \\ 2x^2-2x+2x-2 &= 2x^2-3x+10 && \text{subtract } x^2\text{, combine like terms} \\ -2 &= -3x+10 && \text{add } 3x \\ 3x-2 &= 10 && \text{add } 2 \\ 3x &= 12 && \text{divide by } 3 \\ x &= 4\end{aligned}$$

Our result means that the original rectangle had sides 4 and $2\cdot 4-3=5$. This rectangle has sides $\boxed{\text{4 and 5 units}}$ resulting in an area of 20 unit^2. Let us increase both sides by 1 unit. The new rectangle has sides 5 and 6 units. That is an area of 30 unit^2, and so the increase was indeed 10 square-unit.

21. Solve each of the following formulas.

a) $3x-4y=z$ for x

Solution: Let us look at the formula from the point of view of x. What happened to x was first a multiplication by 3 and then subtraction of $4y$. We will perfom the inverse operations, in a reverse order: add first $4y$ and then divide by 3.

$$\begin{aligned}3x-4y &= z && \text{add } 4y \\ 3x &= z+4y && \text{divide by } 3 \\ x &= \frac{z+4y}{3}\end{aligned}$$

Our solution is $\boxed{x=\dfrac{z+4y}{3}}$.

b) $3x-4y=z$ for y

Solution: The fact that y is apperantly subtracted can be handled in wo different ways.

Method 1. Let us swap the two terms to have $-4y+3x$. Then we can interpret it as multiplication by -4 and then addition of $3x$. To undo those, we wil subtract $3x$ and divide by -4. When simplifying the answer, we need to watch out for the distributive law.

$$\begin{aligned}-4y+3x &= z && \text{subtract } 3x \\ -4y &= z-3x && \text{divide by } -4 \\ y &= \frac{z-3x}{-4}\end{aligned}$$

This result is not simplified because we have a negative sign in the denominator. We can fix this by multiplying both numerator and denominator by -1. With the numerator, we have to apply the distributive law.

$$y=\frac{-1(z-3x)}{-1(-4)}=\frac{-z+3x}{4}$$

Method 2. This is a frequently used trick if we want to avoid division by a negative number. Our first step is to add $4y$ to both sides. Then $4y$ disappears on the left-hand side and appears on the right-hand side. We will solve for y there (or swap the two sides).

$$\begin{aligned} 3x - 4y &= z && \text{add } 4y \\ 3x &= z + 4y && \text{subtract } z \\ 3x - z &= 4y && \text{divide by } 4 \\ \frac{3x - z}{4} &= y \end{aligned}$$

Naturally, our result is the same, $\boxed{y = \dfrac{3x - z}{4}}$.

c) $AB + PQ = S$ for Q

Solution: Let us look at the formula from the point of view of Q. What happened to x was first a multiplication by P and then addition of AB. We will perfom the inverse operations, in a reverse order: subtract AB and then divide by Q.

$$\begin{aligned} AB + PQ &= S && \text{subtract } AB \\ PQ &= S - AB && \text{divide by } P \\ Q &= \frac{S - AB}{P} \end{aligned}$$

Notice that it is essential that P is not zero, because division by zero is not allowed. What if P IS zero?

If P is zero, then the original equation $AB + PQ = S$ becomes $AB + 0 \cdot Q = S$ or $AB = S$ and it is not possible to solve for Q.

