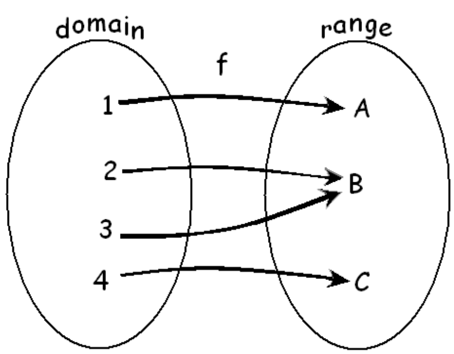
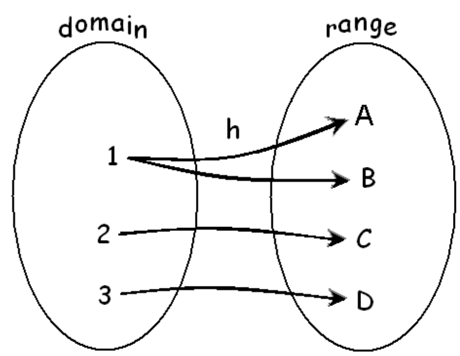
Recall the following definitions of relations and functions.

> **Definition:** A **relation** is an assignment between elements of a non-empty set called domain and another set called the range. We usually denote relations by lower case letters such as f or g.

> **Definition:** A **function** is an assignment between elements of a non-empty set called domain and another set called the range, with one restriction: that to each element of the domain, only one thing can be assigned. We usually denote relations by lower case letters such as f or g.



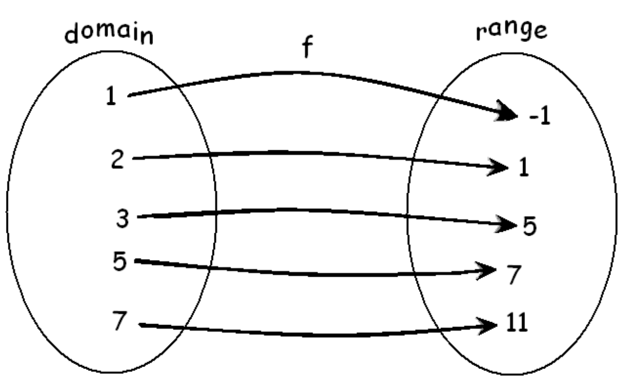


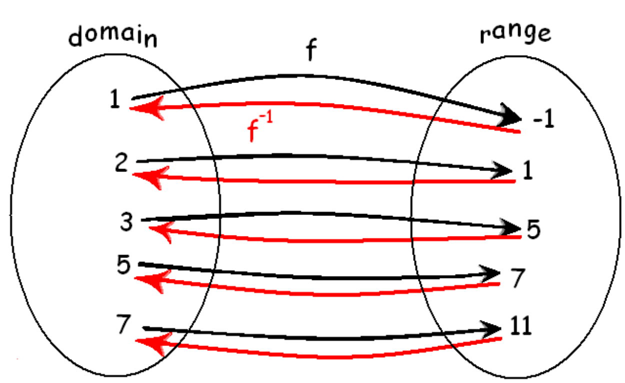
The picture above shows two relations, f and h. While f is a function, h is not, because more than one values are assigned to 1 under h. The distinction between functions and relations that are not functions will be important later.

The main idea about the inverse is the intention of traveling backward along the assignments.

> **Definition:** Suppose that f is a function. The **inverse relation,** denoted by f^{-1} is an assignment that is the reversal of the assignment of f.

Consider the function f shown on the picture below. The inverse relation, denoted by f^{-1} is obtained by reversing the assignments defined by f.





So the statement $f(1) = -1$ can be re-phrased as $f^{-1}(-1) = 1$; the statement $f(2) = 1$ as $f^{-1}(1) = 2$; the statement $f(3) = 5$ as $f^{-1}(5) = 3$, and so on, $f^{-1}(7) = 5$ and $f^{-1}(11) = 7$.

Note 1: It is important to notice that the domain and range of f swap roles under the inverse relation: the domain of f is the range of f^{-1} and the range of f is the domain of f^{-1}. This fact will be useful later.

Note 2: The notation for inverse relation and reciporocal are similar but not quite the same. $f^{-1}(7)$ denotes the inverse relation while $f(7)^{-1}$ denotes reciprocal. So,

$$f^{-1}(7) = 5 \quad \text{and} \quad f(7)^{-1} = \frac{1}{f(7)} = \frac{1}{11}$$

Needless to say, it will be very important for us not to confuse the two notations.

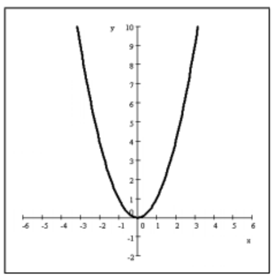
Given the graph of a function f, how do we graph the inverse relation? There are two methods to graph the inverse: by transposing coordinates on the graphs of f and by reflecting the graph of f to the line $y = x$.

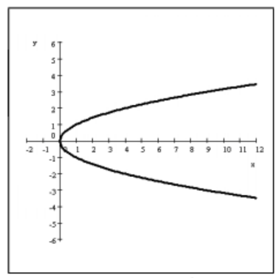
Example 1. Graph the inverse relation f^{-1} for the function $f(x) = x^2$.

Solution: Let us first graph $y = x^2$ first. Since the inverse literally reverses the assignment of f, x becomes y and y becomes x under the inverse relation. So if $(3,9)$ and $(-3,9)$ are points on the graph of f, then switching the two coordinates gives us points on the graph of the inverse. So, $(9,3)$ and $(9,-3)$ are points on the graph of the inverse. We apply this method to more points:

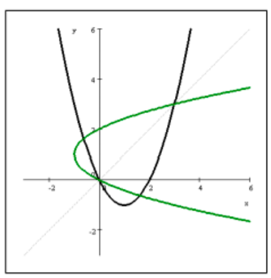
points on f	$(0,0)$	$(1,1)$	$(-1,1)$	$(2,4)$	$(-2,4)$	$(3,9)$	$(-3,9)$	$(4,16)$	$(-4,16)$
points on f^{-1}	$(0,0)$	$(1,1)$	$(1,-1)$	$(4,2)$	$(4,-2)$	$(9,3)$	$(9,-3)$	$(16,4)$	$(16,-4)$

We now plot these points and connect them.


graph of f


graph of f^{-1}

We can graph these two together to better see the symmetry.

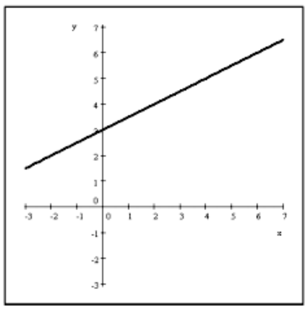


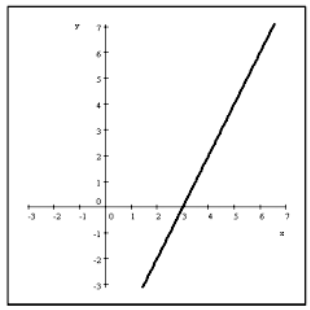
Although it appears that this is a rotation by $90°$, this is not what happens, because the point $(3,9)$ becomes $(9,3)$ on the inverse graph and the point $(-3,9)$ becomes $(9,-3)$. The fact that this is generally not a rotation will be more clear after the next example.

Example 2. Graph the inverse relation g^{-1} for the function $g(x) = \frac{1}{2}x + 3$.

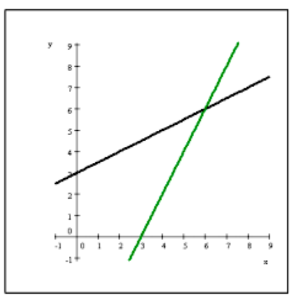
Solution: We will again collect a few points on the graph of g and transpose their coordinates to obtain points on the graph.

points on g	$(-2, 2)$	$\left(-1, \frac{5}{2}\right)$	$(0, 3)$	$\left(1, \frac{7}{2}\right)$	$(2, 4)$	$\left(3, \frac{9}{2}\right)$	$(4, 5)$
points on g^{-1}	$(2, -2)$	$\left(\frac{5}{2}, -1\right)$	$(3, 0)$	$\left(\frac{7}{2}, 1\right)$	$(4, 2)$	$\left(\frac{9}{2}, 3\right)$	$(16, 5)$


graph of g


graph of g^{-1}

We can graph these two together to better see the symmetry:


graphs of g and g^{-1}

Notice that g^{-1} turned out to be a function but not f^{-1}. Since $f(3) = f(-3) = 9$, the inverse relation f^{-1} will fail to be a function because both 3 and -3 will be assigned to 9 under f^{-1}. Indeed, if a function assigns the same $y-$value to two different $x-$values, then the inverse relation will fail to be a function.

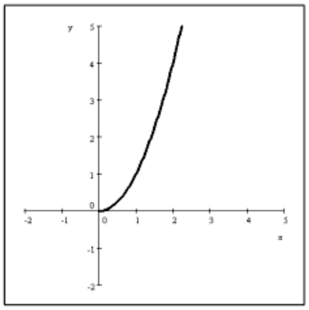
Theorem: Suppose that f is a function. The **inverse relation,** denoted by f^{-1} is a function if and only if the function is one-to-one.

Recall that a function f is one-to-one if to each element of its domain a different element of the range is assigned.

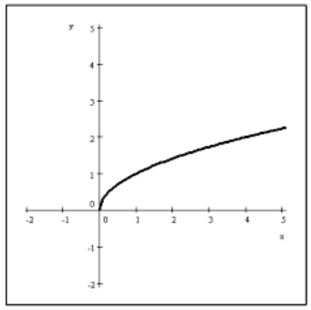
Definition: A function f is one-to-one (or injective) if for all a and b in its domain, if $a \neq b$, then $f(a) \neq f(b)$.

As it turns out, inverse functions are extremely important for mathematicians. Even if a function is not one-to-one, we would still like to define an inverse *function* (and not just a relation), we restrict its domain until it becomes one-to-one and then take the inverse.

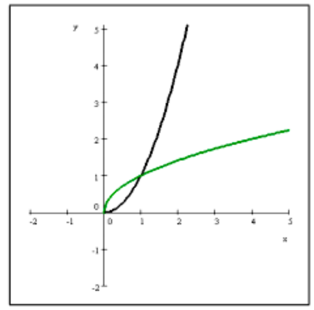
For example, $f(x) = x^2$ is not one-to-one on its full domain, the set of all real numbers. So we restrict its domain to $[0, \infty)$. On this smaller domain, the function is now one-to-one and so the inverse relation is a function.



graph of f



graph of f^{-1}



graphs of f and f^{-1}

This inverse function is the square root function, $f^{-1}(x) = \sqrt{x}$.

Given a function f, how do we compute the formula for the inverse? While f gives an assignment from the $x-$values to the $y-$ values, the inverse establishes an assignment from y back to x. The following is the algorithm to find the formula for the inverse function.

Step 1. Remove the function notation $(f(x))$ and replace it with y.

Step 2. Solve for x in the equation for f.

Step 3. To switch to the inverse, swap x and y in the equation.

Step 4. Remove y and replace it with function notation $(f^{-1}(x))$.

Example 3. Find the equation for the inverse of $f(x) = 2\sqrt[3]{x+1} - 3$

Solution: We will write first $y = 2\sqrt[3]{x+1} - 3$. Then we solve for x in terms of y.

$$
\begin{aligned}
y &= 2\sqrt[3]{x+1} - 3 && \text{add 3} \\
y + 3 &= 2\sqrt[3]{x+1} && \text{divide by 2} \\
\frac{y+3}{2} &= \sqrt[3]{x+1} && \text{raise to the third power} \\
\left(\frac{y+3}{2}\right)^3 &= x + 1 && \text{subtract 1} \\
\left(\frac{y+3}{2}\right)^3 - 1 &= x
\end{aligned}
$$

The assignment of function f is $x = \left(\frac{y+3}{2}\right)^3 - 1$. Under the inverse, x and y are now swapped.

$$\boxed{x = \left(\frac{y+3}{2}\right)^3 - 1 \qquad \text{under } f} \quad \Longrightarrow \quad \boxed{y = \left(\frac{x+3}{2}\right)^3 - 1 \text{ under the inverse, } f^{-1}.}$$

and so the inverse of f is $f^{-1}(x) = \left(\frac{x+3}{2}\right)^3 - 1$.

Note: if we say 'the inverse of f', without specifying whether it is the inverse function or the inverse relation, we always mean the inverse function.

There is one more important and useful property of inverse functions.

Example 4. Suppose that $f(x) = 3x - 8$.

a) Find the inverse relation of f. b) Is f^{-1} a function? c) Compute $f\left(f^{-1}(7)\right)$ and $f^{-1}(f(7))$.

Solution: a) As before, we drop the function notation, solve for x, and swap x and y.

$$\begin{aligned} y &= 3x - 8 \\ y + 8 &= 3x \\ x &= \frac{y+8}{3} \qquad \text{in the inverse, } y = \frac{x+8}{3} \end{aligned}$$

Therefore, $f^{-1}(x) = \dfrac{x+8}{3}$.

b) The formula $x = \dfrac{y+8}{3}$ shows that given y, the value of x is uniquely determined by f. In other words, no two different x 'gets' the same y assigned. Therefore, $f^{-1}(x) = \dfrac{x+8}{3}$ is a function. In such case, we say that f is one-to-one.

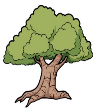
c) $f\left(f^{-1}(7)\right) = f\left(\dfrac{7+8}{3}\right) = f(5) = 3 \cdot 5 - 8 = 7$

d) $f^{-1}(f(7)) = f^{-1}(3 \cdot 7 - 8) = f^{-1}(13) = \dfrac{13+8}{3} = 7$

Theorem: Suppose that f is a one-to-one function. Then f^{-1} is a function such that for all x,

$$f^{-1}(f(x)) = x \quad \text{and} \quad f\left(f^{-1}(x)\right) = x$$

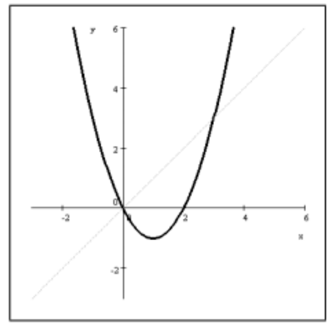
Another way to state this is: $f \circ f^{-1} = id$ and $f^{-1} \circ f = id$ where $id(x) = x$ for all x (the identitiy function).



Enrichment

Suppose that f is not a one-to-one function. So we restrict its domain until it does become one-to-one. Then we take the inverse function. One statement from $f^{-1}(f(x)) = x$ and $f\left(f^{-1}(x)\right) = x$ is still true. Which one? Give an example for how one if these two fails to be true.

Sample Problems



1. The given picture shows the graph of $f(x) = x(x-2)$. Graph the inverse relation in the same coordinate system.

2. Find an equation for the inverse for each of the function given.

a) $f(x) = 3x + 2$

b) $f(x) = (5x-1)^3$

c) $f(x) = \dfrac{x+4}{3x-5}$

d) $f(x) = \log_5(2x-1)$

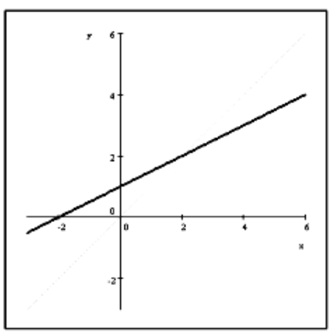
e) $f(x) = e^{5x-1}$

3. Prove that the inverse of a linear function is also linear and the two slopes are reciporcals of each other.

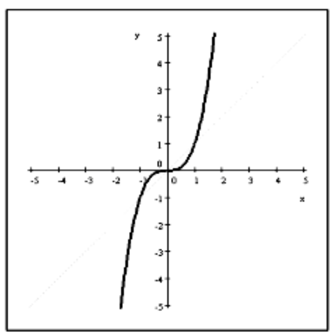
Practice Problems

1. The following pictures show the graphs of functions. In each case, graph the inverse relation.

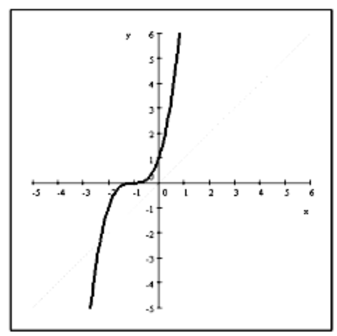
a) $f(x) = \dfrac{1}{2}x + 1$



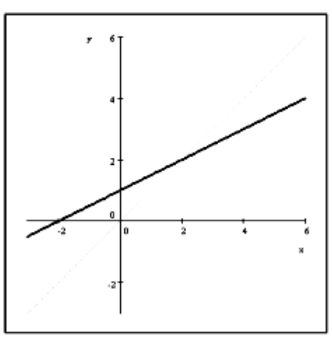
b) $f(x) = x^2 - 2$



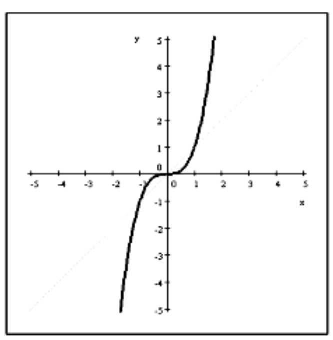
c) $f(x) = x^3$



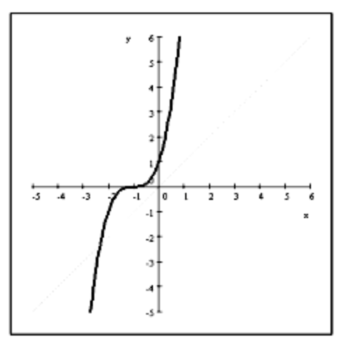
d) $f(x) = 2^x$



e) $f(x) = (x+1)^3$



f) $f(x) = x^2 + 2$



2. Find an equation for the inverse of each of the following functions.

a) $f(x) = 3^{5x-1}$

b) $f(x) = \dfrac{2x-7}{3x+5}$

c) $f(x) = \ln(2x-1)$

d) $f(x) = \dfrac{1}{3x+1}$

e) $f(x) = \dfrac{3\sqrt{x}-5}{2}$

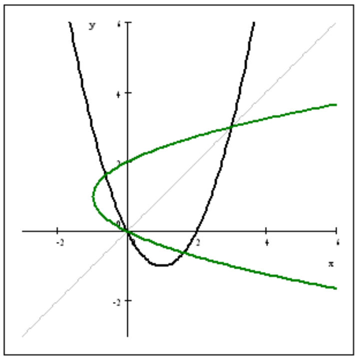
f) $f(x) = \log_3\left(\dfrac{1}{2}x - 7\right)$

g) $f(x) = \dfrac{2x+5}{3x-2}$

Answers

Sample Problems

1. f and f^{-1}



2. a) $f^{-1}(x) = \dfrac{1}{3}(x-2)$

   b) $f^{-1}(x) = \dfrac{1}{5}\left(\sqrt[3]{x}+1\right)$

   c) $f^{-1}(x) = \dfrac{5x+4}{3x-1}$

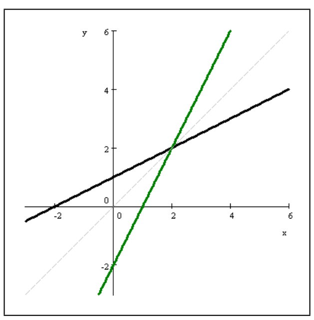
   d) $f^{-1}(x) = \dfrac{1}{2}(5^x+1)$

   e) $f^{-1}(x) = \dfrac{1}{5}(\ln x + 1)$

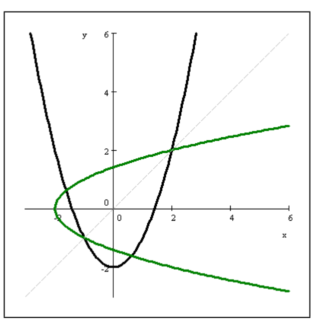
3. see solutions

Practice Problems

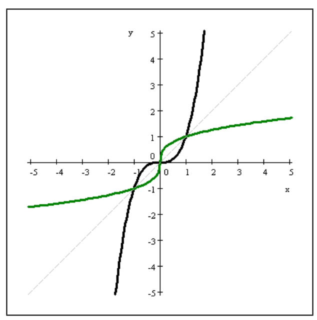
1. a) $f(x) = \dfrac{1}{2}x + 1$



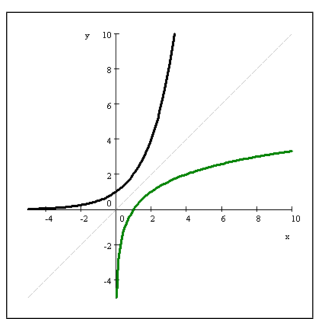
b) $f(x) = x^2 - 2$



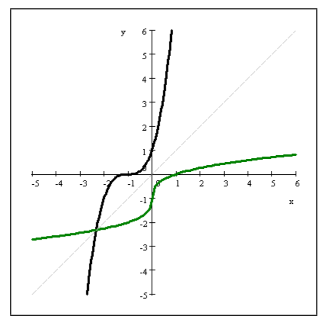
c) $f(x) = x^3$



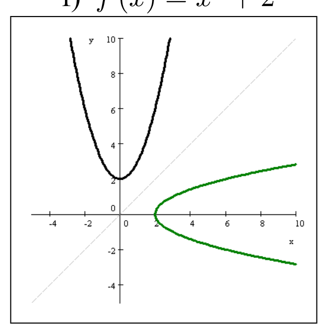
d) $f(x) = 2^x$



e) $f(x) = (x+1)^3$



f) $f(x) = x^2 + 2$



2. a) $f^{-1}(x) = \dfrac{1}{5}(\log_3 x + 1)$

   b) $f^{-1}(x) = \dfrac{5x+7}{-3x+2}$

   c) $f^{-1}(x) = \dfrac{1}{2}(e^x + 1)$

   d) $f^{-1}(x) = \dfrac{-x+1}{3x}$

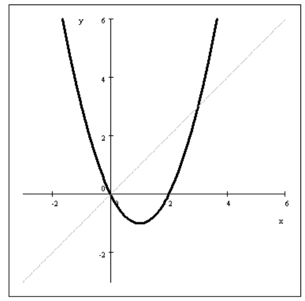
   e) $f^{-1}(x) = \left(\dfrac{2x+5}{3}\right)^2$

   f) $f^{-1}(x) = 2 \cdot 3^x + 14$

   g) $f^{-1}(x) = \dfrac{2x+5}{3x-2}$

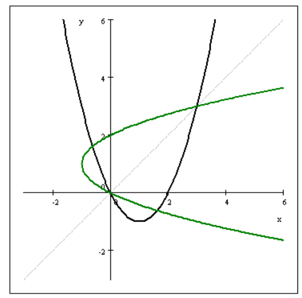
Solutions

1. The given picture shows the graph of $f(x) = x(x-2)$. Graph the inverse relation in the same coordinate system.



Solution: Let us find some points on the function. Substituting easy values for x, we obtain the following points on the graph of f.

$(-2, 8)$, $(-1, 3)$, $(0, 0)$, $(1, -1)$, $(2, 0)$, $(3, 3)$, $(4, 8)$



To obtain points on the graph of the inverse, we simply transpose the coordinates in each points. Thus the following are points on the graph of the inverse:

$(8, -2)$, $(3, -1)$, $(0, 0)$, $(-1, 1)$, $(0, 2)$, $(3, 3)$, $(8, 4)$

We graph these points and connect the points. It is also useful to keep in mind that the graphs should be symmetrical to the line $y = x$.

2. Find an equation for the inverse for each of the function given below.

a) $f(x) = 3x + 2$

Solution: First we drop the function notation and write y instead of $f(x)$. Then we solve for x and finally, swap x and y.

$$\begin{aligned} y &= 3x+2 && \text{subtract 2} \\ y-2 &= 3x && \text{divide by 3} \\ \frac{y-2}{3} &= x && \text{swap } x \text{ and } y \qquad y = \frac{x-2}{3} \qquad f^{-1}(x) = \frac{1}{3}(x-2) \end{aligned}$$

b) $f(x) = (5x-1)^3$

Solution: First we drop the function notation and write y instead of $f(x)$. Then we solve for x and finally, swap x and y.

$$\begin{aligned} y &= (5x-1)^3 && \text{take 3rd root of both sides} \\ \sqrt[3]{y} &= 5x-1 && \text{add 1} \\ \sqrt[3]{y}+1 &= 5x && \text{divide by 5} \\ \frac{\sqrt[3]{y}+1}{5} &= x && \text{swap } x \text{ and } y \qquad y = \frac{1}{5}\left(\sqrt[3]{x}+1\right) \qquad f^{-1}(x) = \frac{1}{5}\left(\sqrt[3]{x}+1\right) \end{aligned}$$

c) $f(x) = \dfrac{x+4}{3x-5}$.

Solution: First we drop the function notation and write y instead of $f(x)$. Then we solve for x and finally, swap x and y.

$$\begin{array}{rcll}
y & = & \dfrac{x+4}{3x-5} & \text{multiply by } 3x-5 \\
y(3x-5) & = & x+4 & \text{distribute} \\
3xy-5y & = & x+4 & \text{add } 5y, \text{ subtract } x \\
3xy-x & = & 5y+4 & \text{factor out } x \\
x(3y-1) & = & 5y+4 & \text{divide by } 3y-1 \\
x & = & \dfrac{5y+4}{3y-1} & \text{swap } x \text{ and } y \quad y = \dfrac{5x+4}{3x-1} \qquad f^{-1}(x) = \dfrac{5x+4}{3x-1}
\end{array}$$

d) $f(x) = \log_5(2x-1)$

Solution: First we drop the function notation and write y instead of $f(x)$. Then we solve for x and finally, swap x and y.

$$\begin{array}{rcll}
y & = & \log_5(2x-1) & \text{re-write it as an exponential statement} \\
5^y & = & 2x-1 & \text{add } 1 \\
5^y+1 & = & 2x & \text{divide by } 2 \\
\dfrac{5^y+1}{2} & = & x \quad \text{swap } x \text{ and } y & y = \dfrac{1}{2}(5^x+1) \qquad f^{-1}(x) = \dfrac{1}{2}(5^x+1)
\end{array}$$

e) $f(x) = e^{5x-1}$

Solution: First we drop the function notation and write y instead of $f(x)$. Then we solve for x and finally, swap x and y.

$$\begin{array}{rcll}
y & = & e^{5x-1} & \text{take the natural logarithm of both sides} \\
\ln y & = & \ln\left(e^{5x-1}\right) & \ln\left(e^{5x-1}\right) = 5x-1 \\
\ln y & = & 5x-1 & \text{add } 1 \\
\ln y+1 & = & 5x & \text{divide by } 5 \\
\dfrac{\ln y+1}{5} & = & x \quad \text{swap } x \text{ and } y & y = \dfrac{1}{5}(\ln x+1) \qquad f^{-1}(x) = \dfrac{1}{5}(\ln x+1)
\end{array}$$

3. Prove that the inverse of a linear function is also linear and the two slopes are reciporcals of each other.
Solution: Let $f(x) = mx+b$. If $m=0$, the function is very badly not on-to-one and the inverse does not exist. If $m \neq 0$, then the inverse is

$$\begin{array}{rcll}
y & = & mx+b & \text{solve for } x \\
y-b & = & mx & m \neq 0 \\
\dfrac{y-b}{m} & = & x & \Longrightarrow \qquad x = \dfrac{1}{m}y - \dfrac{b}{m} \quad \Longrightarrow f^{-1}(x) = \dfrac{1}{m}x - \dfrac{b}{m}
\end{array}$$

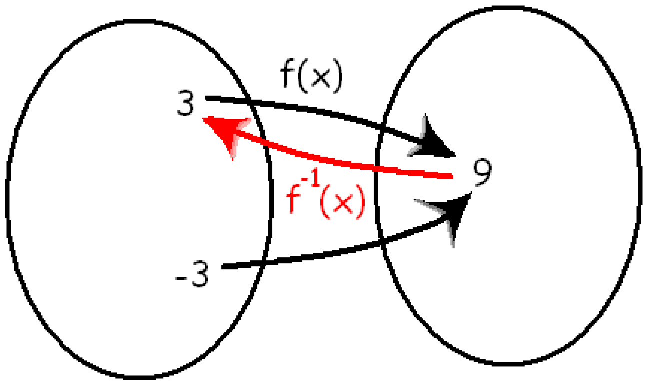
The inverse is a line with slope $\dfrac{1}{m}$.

Solution for the enrichment problem:

Let $f(x) = x^2$. We reduce its domain to $[0, \infty)$, and then the inverse function is $f^{-1}(x) = \sqrt{x}$.

$\left(f\left(f^{-1}(x)\right)\right) = (\sqrt{x})^2 = x$ is still true, but $\left(f^{-1}(f(x))\right) = \sqrt{x^2} = x$ is false.

Consider $\left(f^{-1}(f(-3))\right) = \sqrt{(-3)^2} = \sqrt{9} = 3$. We started with -3 and landed on 3. Therefore, this composition of functions is not the identitiy function. In fact, $\left(f^{-1}(f(-3))\right) = \sqrt{x^2} = |x|$.



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