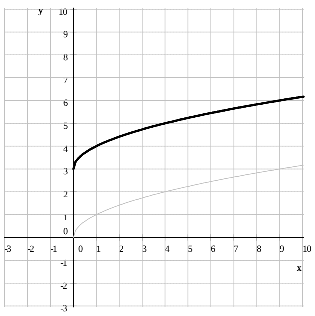
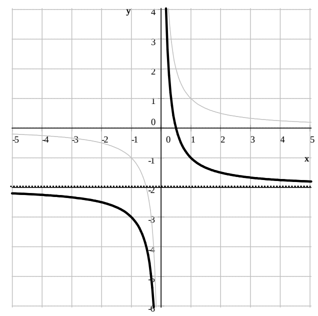
We graph some functions by first graphing a basic function and then applying transformations to the graph. These transformations naturally correspond to operations.

> The graph of $y = f(x) + A$ is obtained by a verical shift applied to the graph of $y = f(x)$ by A units (upward if A is positive, downward if A is negative).

For example, we obtain the graph of $y = \sqrt{x} + 3$ by shifting the graph of $y = \sqrt{x}$ up by three units.



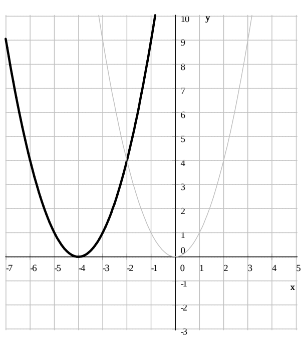
We obtain the graph of $y = \dfrac{1}{x} - 2$ by shifting the graph of $y = \dfrac{1}{x}$ down by two units.



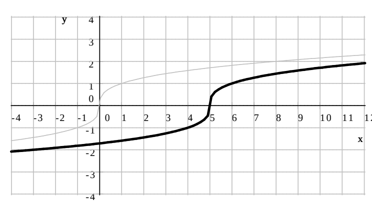
Note: Sometimes it takes a little computation to realize a vertical shift. For example, $\dfrac{1}{x} - 2$ might be presented as $\dfrac{1-2x}{x}$. We need to peform the division to bring the expression to the form where the vertical shift is obvious.

> The graph of $y = f(x + B)$ is obtained by a horizontal shift applied to the graph of $y = f(x)$ by B units (to the left if B is positive, to the right if B is negative).

For example, we obtain the graph of $y = (x+4)^2$ by shifting the graph of $y = x^2$ to the left by four units.



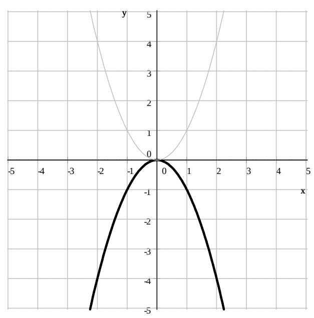
We obtain the graph of $y = \sqrt[3]{x-5}$ by shifting the graph of $y = \sqrt[3]{x}$ to the right by five units.



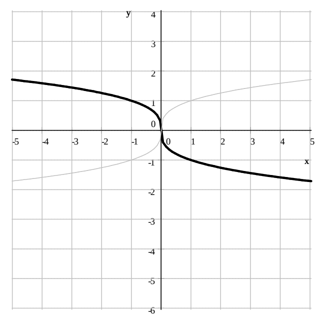
The fact that positive values of B produce a shift to the left (the negative direction) and negative values of B produce a shift to the right (the positive direction) is counter-intuitive first. This is different from the case of the vertical shift. It is important to remember that the directions of horizontal shifts are different from those of the vertical shifts.

The graph of $y = -f(x)$ is obtained by reflecting the graph of $y = f(x)$ to the $x-$ axis.

For example, we obtain the graph of $y = -x^2$ by reflecting the graph of $y = x^2$ to the $x-$axis.

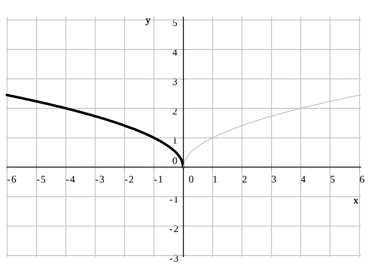


We obtain the graph of $y = -\sqrt[3]{x}$ by shifting by reflecting the graph of $y = \sqrt[3]{x}$ to the $x-$axis.

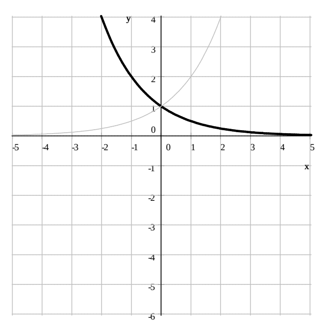


The graph of $y = f(-x)$ is obtained by reflecting the graph of $y = f(x)$ to the $y-$ axis.

For example, we obtain the graph of $y = \sqrt{-x}$ by reflecting the graph of $y = \sqrt{x}$ to the $y-$axis.

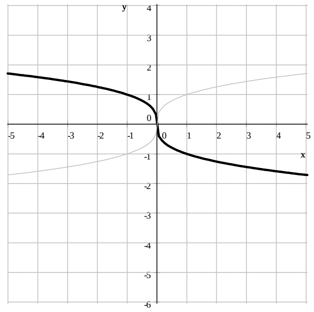


We obtain the graph of $y = 2^{-x}$ by reflecting the graph of $y = 2^x$ to the $y-$axis.

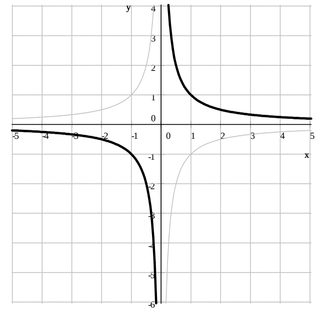


Please note that sometimes two different transformations produce the same affect. For example, reflecting $y = x^3$ to the $x-$axis and to the $y-$axis produces the same affect, because $\sqrt[3]{(-x)^3} = \sqrt[3]{-x^3} = -\sqrt[3]{x^3}$.

The graphs of $y = -\sqrt[3]{x}$ and $y = \sqrt[3]{-x}$ are the same .

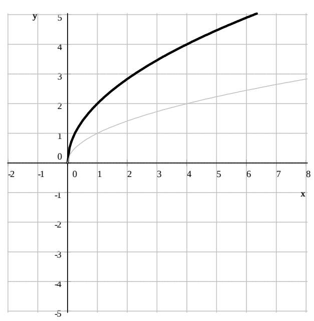


The graphs of $y = -\dfrac{1}{x}$ and $y = \dfrac{1}{-x}$. are the same.

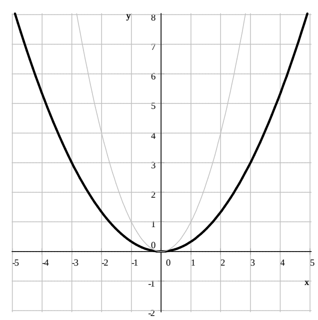


Let $c > 0$. The graph of $y = cf(x)$ is obtained by stretching the graph of $y = f(x)$ by a factor of c in the $y-$ direction.

For example, we obtain the graph of $y = 2\sqrt{x}$ by stretching the graph of $y = \sqrt{x}$ by a factor of 2 in the $y-$direction.



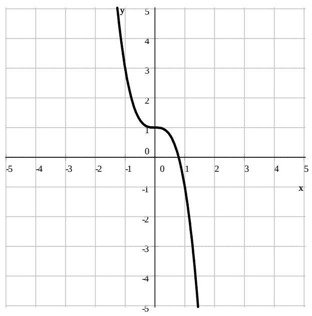
We obtain the graph of $y = \dfrac{1}{3}x^2$ by stretching the graph of $y = x^2$ by a factor of $\dfrac{1}{3}$ in the $y-$direction.



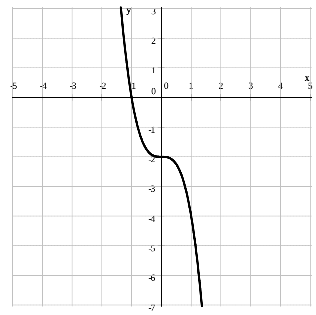
The order of these transformations is important. Consider for example, $y = -2x^3 + 1$ and $y = -2\left(x^3 + 1\right)$. We should always apply the transformations in the same order we would perform the indicated operations. In the case of $y = -2x^3 + 1$, the vertical shift is the last transformation, while in the case of $y = -2\left(x^3 + 1\right)$, the vertical shift by 1 unit is performed before the reflection and stretching. Of course, another option is to first simplify the expression $y = -2\left(x^3 + 1\right) = -2x^3 - 2$ and then performing the transformations accordingly.

Please also note that the order of transformations does not always matter. For example, in the case of $y = -2x^3$ the reflection to the $y-$axis and the stretching along the $y-$axis are commutative and they can be performed in any order.

To graph $y = -2x^3 + 1$
We start with the graph of $y = x^3$.
We reflect the graph on the $x-$axis and stretch the graph by a factor of 2 in the $y-$direction. Finally, we shift the graph up by 1 unit.



To graph $y = -2\left(x^3 + 1\right)$
We start with the graph of $y = x^3$.
We shift the graph up by 1 unit.
We reflect the graph on the $x-$axis and stretch the graph by a factor of 2 in the $y-$direction.



Sample Problems

1. Graph each of the following functions. In each case, describe what basic functions and transformations you would use to graph the function.

   a) $f(x) = \dfrac{1}{2}|x-4| - 5$ b) $f(x) = -2\sqrt{3-x} + 4$ c) $f(x) = -\dfrac{1}{x+3} + 2$

2. Graph each of the following functions. In each case, describe what basic functions and transformations you would use to graph the function.

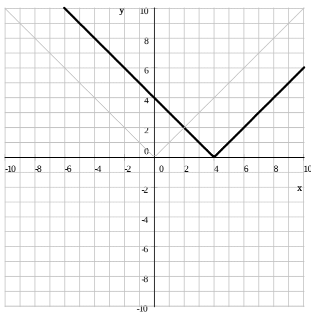
   a) $f(x) = \dfrac{2}{x-4} + 1$ b) $f(x) = 2\sqrt{-x+1}$ c) $f(x) = \dfrac{1}{3}\left(2^{x-2}\right) - 1$

Sample Problems - Solutions

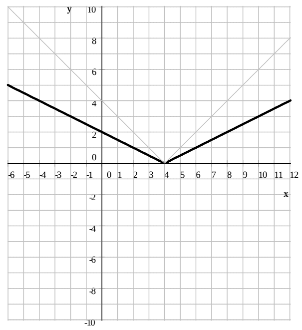
1. a) $f(x) = \dfrac{1}{2}|x-4| - 5$

Start with $y = |x|$

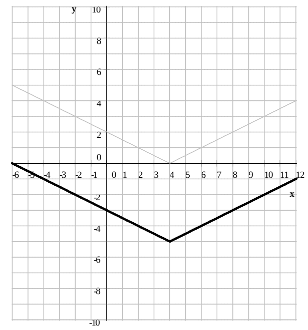
Shift to the right by 4 units



Stretch along the $y-$axis by a factor of $\dfrac{1}{2}$



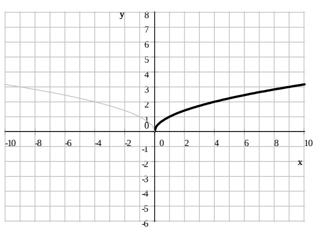
Shift down by 5 units



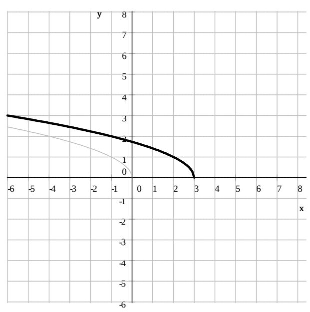
b) $f(x) = -2\sqrt{3-x} + 4 = -2\sqrt{-x+3} + 4$

Start with $y = \sqrt{x}$

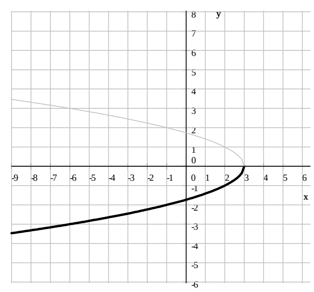
Reflect to $y-$axis



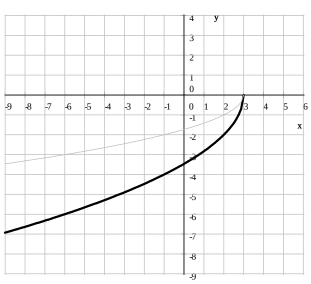
Shift to the right by 3units



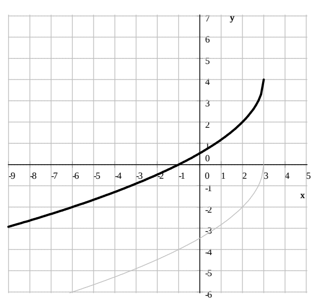
Reflect to $y-$axis



Stretch by a factor of 2



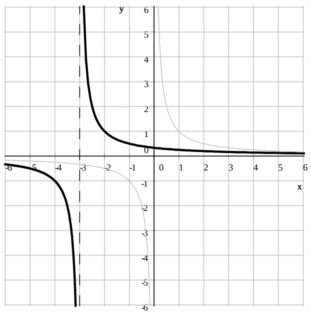
Shift up by 4 units



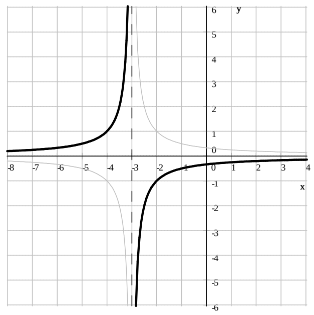
c) $f(x) = -\dfrac{1}{x+3} + 2$

start with $y = \dfrac{1}{x}$

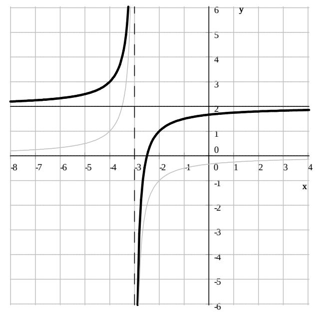
shift to the left by 3 units



reflect to the $y-$axis



shift up by 2 units



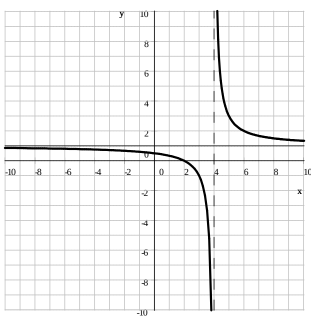
2. a) $f(x) = \dfrac{2}{x-4} + 1$

start with $y = \dfrac{1}{x}$

shift to the right by 4 units

stretch by 2 along the $y-$axis

shift up by 1 unit



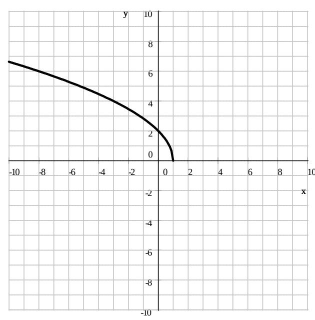
b) $f(x) = 2\sqrt{-x+1}$

start with $y = \sqrt{x}$

reflect to $y-$axis

shift to the left by 1 unit

stretch along the $y-$axis by 2



c) $f(x) = \dfrac{1}{3}\left(2^{x-2}\right) - 1$

start with $y = 2^x$

shift to the right by 2 units

stretch by a factor of $\dfrac{1}{3}$

shift down by 1 unit