

## Sample Problems

Prove each of the following statements by induction.

1. For all natural numbers  $n$ ,
  - a)  $1 + 2 + 3 + \dots + n = \frac{n(n+1)}{2}$ .
  - b)  $1^2 + 2^2 + 3^2 + \dots + n^2 = \frac{n(n+1)(2n+1)}{6}$ .
  - c)  $1^3 + 2^3 + 3^3 + \dots + n^3 = \frac{n^2(n+1)^2}{4}$ .
2. For all natural numbers  $n$ ,  $\frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} + \frac{1}{3 \cdot 4} + \dots + \frac{1}{n(n+1)} = \frac{n}{n+1}$ .
3. For all natural numbers  $n$ ,  $1 \cdot 2 + 2 \cdot 2^2 + 3 \cdot 2^3 + 4 \cdot 2^4 + \dots + n \cdot 2^n = 2(1 + (n-1)2^n)$ .
4. For all natural numbers  $n \geq 5$ ,  $2^n \geq n^2$ .
5. Prove that for all natural number  $n$ ,  $10^n$  can be written as a sum of two perfect squares.
6. Recall  $(F_n)$  is the Fibonacci sequence, defined as

$$F_1 = 1, \quad F_2 = 1, \quad \text{and} \quad F_{n+2} = F_n + F_{n+1}$$

Prove each of the following statements for all natural numbers  $n$ .

- a)  $F_1 + F_2 + F_3 + \dots + F_n = F_{n+2} - 1$
- b)  $F_1^2 + F_2^2 + F_3^2 + \dots + F_n^2 = F_n F_{n+1}$
- c)  $F_1 + F_3 + \dots + F_{2n-1} = F_{2n}$

## Sample Problems - Solutions

1. a) For all natural numbers  $n$ ,  $1 + 2 + 3 + \dots + n = \frac{n(n+1)}{2}$ .

Proof:

Part 1: We check for the first few values of  $n$ .

If  $n = 1$ , then

$$\text{LHS} = 1 = 1 \quad \text{and} \quad \text{RHS} = \frac{1(1+1)}{2} = \frac{2}{2} = 1$$

If  $n = 2$ , then

$$\text{LHS} = 1 + 2 = 3 \quad \text{and} \quad \text{RHS} = \frac{2(2+1)}{2} = \frac{6}{2} = 3$$

If  $n = 3$ , then

$$\text{LHS} = 1 + 2 + 3 = 6 \quad \text{and} \quad \text{RHS} = \frac{3(3+1)}{2} = \frac{12}{2} = 6$$

So the statement is true for  $n = 1, 2$  and  $3$ .

Part 2. Suppose that  $k$  is a positive integer for which

$$1 + 2 + 3 + \dots + k = \frac{k(k+1)}{2} \quad \text{This is the inductual hypotheses}$$

Let us add  $k + 1$  to both sides. The left-hand side becomes

$$1 + 2 + 3 + \dots + k + (k + 1)$$

and the right hand side becomes

$$\begin{aligned} \text{RHS} &= \frac{k(k+1)}{2} + (k+1) = \frac{k(k+1)}{2} + \frac{2(k+1)}{2} = \frac{k(k+1) + 2(k+1)}{2} \\ &= \frac{k^2 + k + 2k + 2}{2} = \frac{k^2 + 3k + 2}{2} = \frac{(k+1)(k+2)}{2} = \frac{(k+1)((k+1)+1)}{2} \end{aligned}$$

So we have proved that for all positive integers  $k$ , if

$$1 + 2 + 3 + \dots + k = \frac{k(k+1)}{2} \quad \text{is true}$$

then

$$1 + 2 + 3 + \dots + k + (k + 1) = \frac{(k + 1)((k + 1) + 1)}{2}$$

which completes our proof.

b) For all natural numbers  $n$ ,  $1^2 + 2^2 + 3^2 + \dots + n^2 = \frac{n(n+1)(2n+1)}{6}$ .

Proof:

Part 1: We check for the first few values of  $n$ .

If  $n = 1$ , then

$$\text{LHS} = 1^2 = 1 \quad \text{and} \quad \text{RHS} = \frac{1(1+1)(2 \cdot 1 + 1)}{6} = \frac{6}{6} = 1$$

If  $n = 2$ , then

$$\text{LHS} = 1^2 + 2^2 = 5 \quad \text{and} \quad \text{RHS} = \frac{2(2+1)(2 \cdot 2 + 1)}{6} = \frac{30}{6} = 5$$

If  $n = 3$ , then

$$\text{LHS} = 1^2 + 2^2 + 3^2 = 14 \quad \text{and} \quad \text{RHS} = \frac{3(3+1)(2 \cdot 3 + 1)}{6} = \frac{84}{6} = 14$$

So the statement is true for  $n = 1, 2$  and  $3$ .

Part 2. Suppose that  $k$  is a positive integer for which

$$1^2 + 2^2 + 3^2 + \dots + k^2 = \frac{k(k+1)(2k+1)}{6} \quad \text{This is the inductual hypotheses}$$

Let us add  $(k+1)^2$  to both sides. The left-hand side becomes

$$1^2 + 2^2 + 3^2 + \dots + k^2 + (k+1)^2$$

and the right hand side becomes

$$\text{RHS} = \frac{k(k+1)(2k+1)}{6} + (k+1)^2 = \frac{k(k+1)(2k+1)}{6} + \frac{6(k+1)^2}{6}$$

We factor out  $\frac{k+1}{6}$

$$\begin{aligned} \text{RHS} &= \frac{k(k+1)(2k+1)}{6} + \frac{6(k+1)^2}{6} = \frac{k+1}{6} (k(2k+1) + 6(k+1)) \\ &= \frac{k+1}{6} (2k^2 + k + 6k + 6) = \frac{k+1}{6} (2k^2 + 7k + 6) \end{aligned}$$

and so happens  $2k^2 + 7k + 6$  factors as  $(k+2)(2k+3)$ . So we have

$$\begin{aligned} \text{RHS} &= \frac{k+1}{6} (2k^2 + 7k + 6) = \frac{k+1}{6} (k+2)(2k+3) = \frac{(k+1)(k+2)(2k+3)}{6} \\ &= \frac{(k+1)((k+1)+1)(2(k+1)+1)}{6} \end{aligned}$$

So we have proved that for all positive integers  $k$ , if

$$1^2 + 2^2 + 3^2 + \dots + k^2 = \frac{k(k+1)(2k+1)}{6}$$

then

$$1^2 + 2^2 + 3^2 + \dots + k^2 + (k+1)^2 = \frac{(k+1)((k+1)+1)(2(k+1)+1)}{6}$$

which completes our proof.

c) For all natural numbers  $n$ ,  $1^3 + 2^3 + 3^3 + \dots + n^3 = \frac{n^2(n+1)^2}{4}$

Proof:

Part 1: We check for the first few values of  $n$ .

If  $n = 1$ , then

$$\text{LHS} = 1^3 = 1 \quad \text{and} \quad \text{RHS} = \frac{1^2(1+1)^2}{4} = \frac{4}{4} = 1$$

If  $n = 2$ , then

$$\text{LHS} = 1^3 + 2^3 = 9 \quad \text{and} \quad \text{RHS} = \frac{2^2(2+1)^2}{4} = \frac{36}{4} = 9$$

If  $n = 3$ , then

$$\text{LHS} = 1^3 + 2^3 + 3^3 = 36 \quad \text{and} \quad \text{RHS} = \frac{3^2(3+1)^2}{2^2} = \frac{144}{4} = 36$$

So the statement is true for  $n = 1, 2$  and  $3$ .

Part 2. Suppose that  $k$  is a positive integer for which

$$1^3 + 2^3 + 3^3 + \dots + k^3 = \frac{k^2(k+1)^2}{4} \quad \text{This is the inductual hypotheses}$$

Let us add  $(k+1)^3$  to both sides. The left-hand side becomes

$$1^3 + 2^3 + 3^3 + \dots + k^3 + (k+1)^3$$

and the right hand side becomes

$$\begin{aligned} \text{RHS} &= \frac{k^2(k+1)^2}{4} + (k+1)^3 = \frac{k^2(k+1)^2}{4} + \frac{4(k+1)^3}{4} \quad \text{factor out } \frac{(k+1)^2}{4} \\ &= \frac{(k+1)^2}{4} (k^2 + 4(k+1)) = \frac{(k+1)^2}{4} (k^2 + 4k + 4) = \frac{(k+1)^2}{4} (k+2)^2 = \frac{(k+1)^2(k+1+1)^2}{4} \end{aligned}$$

So we have proved that for all positive integers  $k$ , if

$$1^3 + 2^3 + 3^3 + \dots + k^3 = \frac{k^2(k+1)^2}{4} \quad \text{is true}$$

then

$$1^3 + 2^3 + 3^3 + \dots + k^3 + (k+1)^3 = \frac{(k+1)^2((k+1)+1)^2}{4}$$

which completes our proof.

2. For all natural numbers  $n$ ,  $\frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} + \frac{1}{3 \cdot 4} + \dots + \frac{1}{n(n+1)} = \frac{n}{n+1}$ .

Proof: Induction on  $n$ .

Part 1. If  $n = 1$ , then

$$\text{LHS} = \frac{1}{1 \cdot 2} = \frac{1}{2} \quad \text{and} \quad \text{RHS} = \frac{1}{1+1} = \frac{1}{2}$$

If  $n = 2$ , then

$$\text{LHS} = \frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} = \frac{1}{2} + \frac{1}{6} = \frac{2}{3} \quad \text{and} \quad \text{RHS} = \frac{2}{2+1} = \frac{2}{3}$$

Part 2. Suppose that  $k$  is a natural number such that

$$\frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} + \frac{1}{3 \cdot 4} + \dots + \frac{1}{k(k+1)} = \frac{k}{k+1} \quad (\text{Induction hypothesis})$$

We will add  $\frac{1}{(k+1)(k+1+1)} = \frac{1}{(k+1)(k+2)}$  to both sides.

$$\frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} + \frac{1}{3 \cdot 4} + \dots + \frac{1}{k(k+1)} + \frac{1}{(k+1)(k+2)} = \frac{k}{k+1} + \frac{1}{(k+1)(k+2)}$$

Let us simplify the right hand side

$$\begin{aligned} \frac{k}{k+1} + \frac{1}{(k+1)(k+2)} &= \frac{k(k+2)}{(k+1)(k+2)} + \frac{1}{(k+1)(k+2)} = \frac{k(k+2)+1}{(k+1)(k+2)} \\ &= \frac{k^2+2k+1}{(k+1)(k+2)} = \frac{(k+1)^2}{(k+1)(k+2)} = \frac{k+1}{k+2} \end{aligned}$$

Thus we have that from the induction hypotheses

$$\frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} + \frac{1}{3 \cdot 4} + \dots + \frac{1}{k(k+1)} = \frac{k}{k+1}$$

the statement

$$\frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} + \frac{1}{3 \cdot 4} + \dots + \frac{1}{(k+1)(k+2)} = \frac{k+1}{k+2}$$

follows. This completes our proof.

3. For all natural numbers  $n$ ,  $1 \cdot 2 + 2 \cdot 2^2 + 3 \cdot 2^3 + 4 \cdot 2^4 + \dots + n \cdot 2^n = 2(1 + (n-1)2^n)$ .

Proof:

Part 1. If  $n = 1$ , then

$$\text{LHS} = 1 \cdot 2 = 2 \quad \text{and} \quad \text{RHS} = 2(1 + (1-1)2^1) = 2 \cdot 1 = 2$$

If  $n = 2$ , then

$$\text{LHS} = 1 \cdot 2 + 2 \cdot 2^2 = 10 \quad \text{and} \quad \text{RHS} = 2(1 + (2-1)2^2) = 2 \cdot 5 = 10$$

Part 2. Suppose that  $k$  is a natural number such that

$$1 \cdot 2 + 2 \cdot 2^2 + 3 \cdot 2^3 + 4 \cdot 2^4 + \dots + k \cdot 2^k = 2(1 + (k-1)2^k) \quad (\text{Induction hypothesis})$$

Let us add  $(k+1)2^{k+1}$  to both sides. The left-hand side becomes

$$\text{LHS} = 1 \cdot 2 + 2 \cdot 2^2 + 3 \cdot 2^3 + 4 \cdot 2^4 + \dots + k \cdot 2^k + (k+1)2^{k+1}$$

and the right-hand side becomes

$$\begin{aligned} \text{RHS} &= 2 \left( 1 + (k-1)2^k \right) + (k+1)2^{k+1} = 2 \left( 1 + (k-1)2^k \right) + (k+1)2^k \cdot 2 && \text{factor out 2} \\ &= 2 \left( 1 + (k-1)2^k + (k+1)2^k \right) = 2 \left( 1 + k2^k - 2^k + k2^k + 2^k \right) = \\ &= 2 \left( 1 + 2k2^k \right) = 2 \left( 1 + k2^{k+1} \right) \end{aligned}$$

which completes our proof.

4. For all natural numbers  $n \geq 5$ ,  $2^n \geq n^2$

Note: this is a very interesting example illustrating how induction works, how we need both parts to work together to form a logically sound proof. The statement seems to be true immediately at  $n = 1$  but then surprisingly, it will be false for a few values of  $n$ .

Proof:

Part 1. If  $n = 1$ , then

$$\text{LHS} = 2^1 = 2 \quad \text{and} \quad \text{RHS} = 1^2 = 1 \quad \text{and} \quad 2 > 1 \quad \text{is true}$$

If  $n = 2$ , then

$$\text{LHS} = 2^2 = 4 \quad \text{and} \quad \text{RHS} = 2^2 = 4 \quad \text{and} \quad 4 > 4 \quad \text{is false!}$$

If  $n = 3$ , then

$$\text{LHS} = 2^3 = 8 \quad \text{and} \quad \text{RHS} = 3^2 = 9 \quad \text{and} \quad 8 > 9 \quad \text{is false!}$$

If  $n = 4$ , then

$$\text{LHS} = 2^4 = 16 \quad \text{and} \quad \text{RHS} = 4^2 = 16 \quad \text{and} \quad 16 > 16 \quad \text{is false!}$$

If  $n = 5$ , then

$$\text{LHS} = 2^5 = 32 \quad \text{and} \quad \text{RHS} = 5^2 = 25 \quad \text{and} \quad 32 > 25 \quad \text{is true}$$

If  $n = 6$ , then

$$\text{LHS} = 2^6 = 64 \quad \text{and} \quad \text{RHS} = 6^2 = 36 \quad \text{and} \quad 64 > 36 \quad \text{is true}$$

At this point we have a sense that the statement will stay true because the left-hand side doubles when we go from  $k$  to  $k + 1$  while the right-hand side just increases from  $k^2$  to  $(k + 1)^2$ . The proof in part 2 will formalize this idea, that between the two types of growth, doubling is much 'faster'.

All we need to show is that moving from  $k^2$  to  $(k + 1)^2$  is a smaller increment than doubling. In short, that

$$\begin{aligned} (k+1)^2 &< 2k^2 \\ k^2 + 2k + 1 &< 2k^2 && \text{subtract } k^2 \\ 2k + 1 &< k^2 \end{aligned}$$

This should be easy to prove for most positive integers. We can either solve the quadratic inequality or be a little bit sloppy or generous and say that if  $k$  is greater than 1, then

$$\begin{aligned} 1 &< k && \text{add } 2k \\ 2k + 1 &< 2k + k \\ 2k + 1 &< 3k \end{aligned}$$

and if  $k$  is greater than 3, then

$$\begin{aligned} 3 &< k && \text{multiply by } k > 0 \\ 3k &< k^2 \end{aligned}$$

In short, if  $k > 3$ , then  $2k + 1 < 3k < k^2$  and so

$$\begin{aligned} 2k + 1 &< k^2 && \text{add } k^2 \text{ to both sides} \\ k^2 + 2k + 1 &< 2k^2 \\ (k + 1)^2 &< 2k^2 \end{aligned}$$

This will be the core of the proof in part 2.

Part 2. Suppose that  $k$  is a natural number such that

$$2^k > k^2 \quad (\text{Induction hypothesis})$$

Let us multiply both sides by 2. The left-hand side becomes

$$\text{LHS} = 2 \cdot 2^k = 2^{k+1}$$

and the right-hand side becomes

$$\text{RHS} = 2k^2$$

So we have

$$\begin{aligned} 2^k &> k^2 \\ 2^{k+1} &> 2k^2 \end{aligned}$$

If  $k > 3$ , then

$$2^{k+1} > 2k^2 = k^2 + k^2 > k^2 + 3k > k^2 + 2k + 1 = (k + 1)^2$$

and so we have that if  $2^k > k^2$  is true AND  $k > 3$ , then  $2^{k+1} > (k + 1)^2$ .

So we could easily prove the 'inheritance' property for all integers greater than 3 but the statement itself is NOT true for  $n = 3$ . Both components of the proof work together starting at  $n = 5$ .

	statement	inheritance to the next number
$n = 1$	true	false
$n = 2$	false	false
$n = 3$	false	true (but we didn't prove it)
$n = 4$	false	true
$n = 5$	true	true

This table illustrates why induction only works here for  $n \geq 5$ .