Sample Problems

1. Consider the expression $(x+1)\,x\,(x-3)$.
   a) Find all values of x for which this expression is 0.
   b) Find all values of x for which this expression is positive.
   c) Find all values of x for which this expression is negative.
   d) Based on the previous parts, sketch the graph of $f(x) = (x+1)\,x\,(x-3)$.

2. Consider the expression $-3(x+4)(x-2)^2$.
   a) Find all values of x for which this expression is 0.
   b) Find all values of x for which this expression is positive.
   c) Find all values of x for which this expression is negative.
   d) Based on the previous parts, sketch the graph of $f(x) = -3(x+4)(x-2)^2$.

3. Consider the expression $5x - x^3$.
   a) Find all values of x for which this expression is 0.
   b) Find all values of x for which this expression is positive.
   c) Find all values of x for which this expression is negative.
   d) Based on the previous parts, sketch the graph of $f(x) = 5x - x^3$.

4. Consider the expression $(x+2)(x-1)^2(x-3)^3$.
   a) Find all values of x for which this expression is 0.
   b) Find all values of x for which this expression is positive.
   c) Find all values of x for which this expression is negative.
   d) Based on the previous parts, sketch the graph of $f(x) = (x+2)(x-1)^2(x-3)^3$.

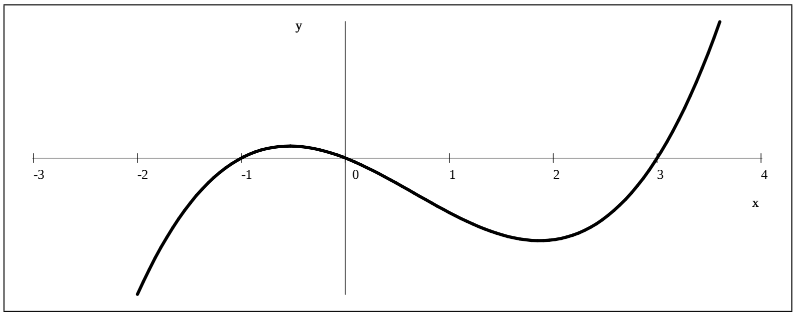
Practice Problems

Sketch the graph of each of the following functions given.

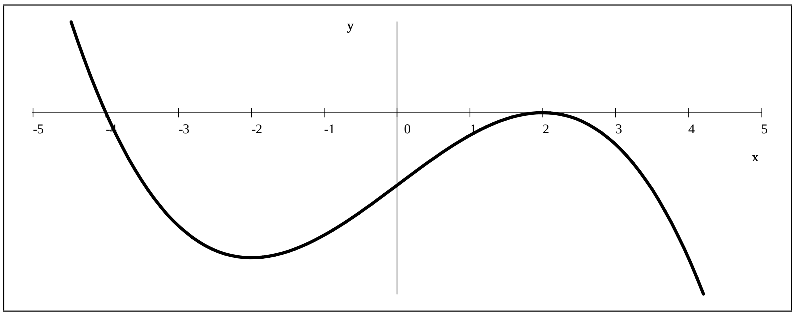
1. $f(x) = (x+3)\,x\,(x-2)$
2. $f(x) = 9x - x^3$
3. $f(x) = x^3 - 2x^2 - 8x$
4. $f(x) = \left(x - x^2 + 2\right)(x-4)$
5. $f(x) = x^3 - 4x$
6. $f(x) = -2(x+1)(x-3)^2$
7. $f(x) = (x+2)^2\,x\,(x-3)$
8. $f(x) = x^5 - x^3$
9. $f(x) = -\left(x^2-4\right)^2$
10. $f(x) = -(x+2)\,x^2\,(x-3)^3$
11. $f(x) = \left(x^2-6\right)(x-1)^2(x+1)$
12. $f(x) = x^5 - 10x^3$
13. $f(x) = -\frac{1}{6}\left(x^2-9\right)\left(x^2-1\right)$
14. $f(x) = \left(x^2-25\right)\left(x^2-5\right)$
15. $f(x) = x^2 - x^4$

Sample Problems - Answers

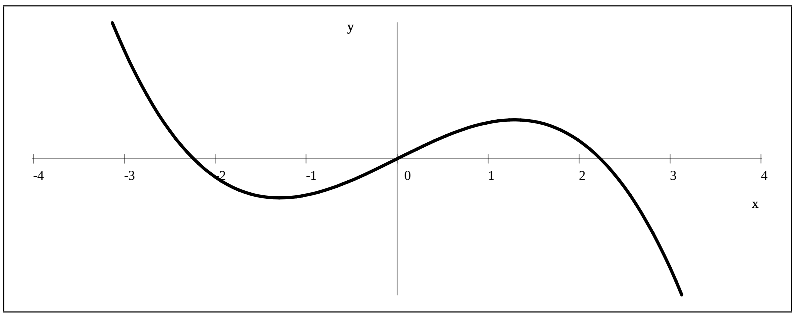
1.) a) $x = -1$, $x = 0$, and $x = 3$ b) $(-1, 0) \cup (3, \infty)$ c) $(-\infty, -1) \cup (0, 3)$ d) see below



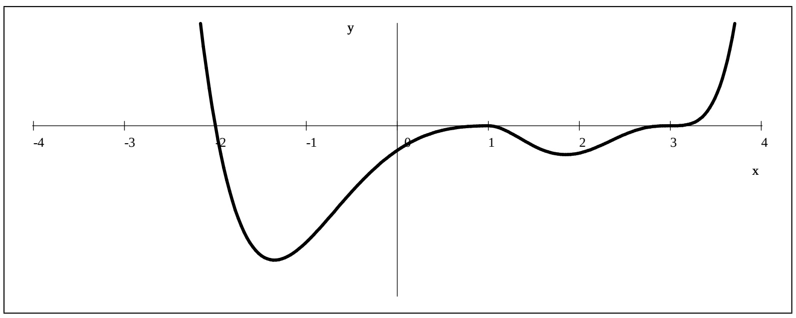
2.) a) $x = -4$ and $x = 2$ b) $(-\infty, -4)$ c) $(-4, 2) \cup (2, \infty)$ d) see below



3.) a) $x = -\sqrt{5}, 0, \sqrt{5}$ b) $\left(-\infty, -\sqrt{5}\right) \cup \left(0, \sqrt{5}\right)$ c) $\left(-\sqrt{5}, 0\right) \cup \left(\sqrt{5}, \infty\right)$ d) see below

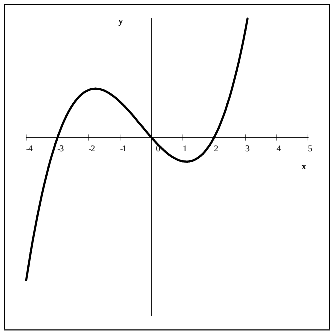


4.) a) $x = -2, 1, 3$ b) $(-\infty, -2) \cup (3, \infty)$ c) $(-2, 1) \cup (1, 3)$ d) see below

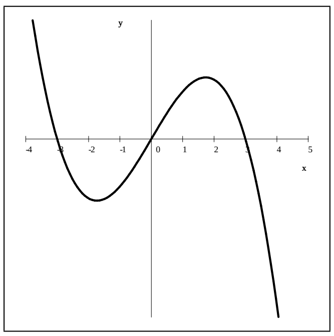


Practice Problems - Answers

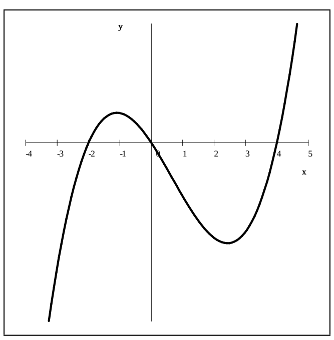
1. $f(x) = (x+3)\,x\,(x-2)$



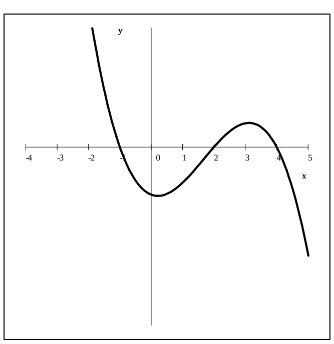
2. $f(x) = 9x - x^3 = -(x+3)\,x\,(x-3)$



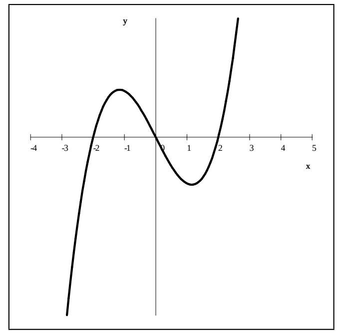
3. $f(x) = x^3 - 2x^2 - 8x = (x+2)\,x\,(x-4)$



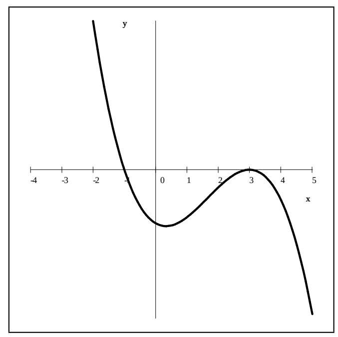
4. $f(x) = \left(x - x^2 + 2\right)(x-4)$
$= -(x+1)(x-2)(x-4)$



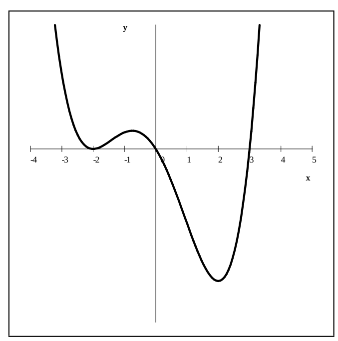
5. $f(x) = x^3 - 4x = (x+2)\,x\,(x-2)$



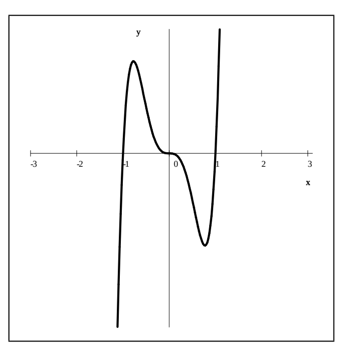
6. $f(x) = -2(x+1)(x-3)^2$



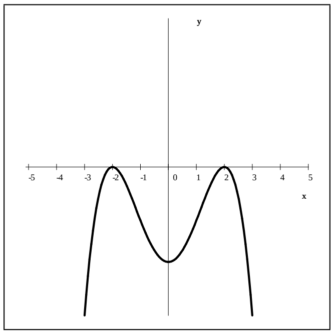
7. $f(x) = (x+2)^2\,x\,(x-3)$



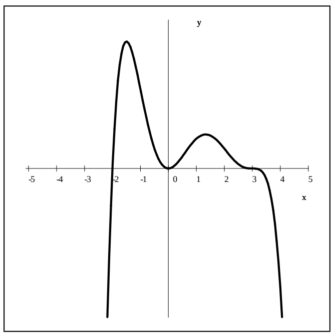
8. $f(x) = x^5 - x^3 = (x+1)\,x^3\,(x-1)$



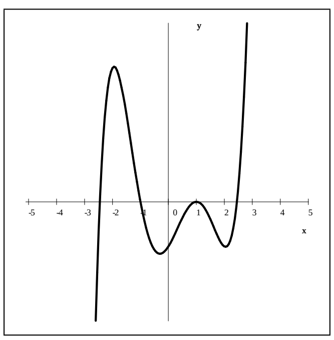
9. $f(x) = -\left(x^2-4\right)^2 = -(x+2)^2(x-2)^2$



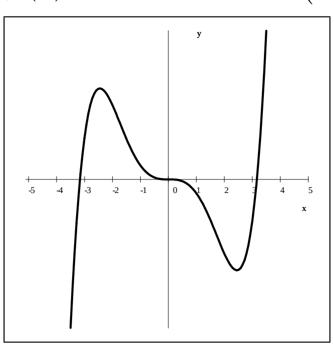
10. $f(x) = -(x+2)\,x^2\,(x-3)^3$



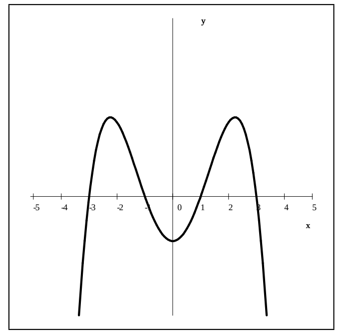
11. $f(x) = \left(x^2-6\right)(x-1)^2(x+1)$
$= \left(x+\sqrt{6}\right)(x+1)(x-1)^2\left(x-\sqrt{6}\right)$



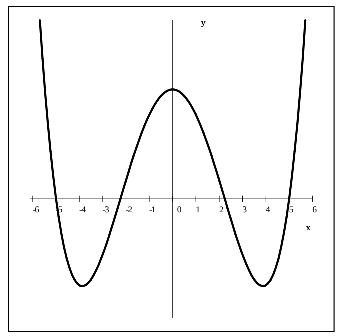
12. $f(x) = x^5 - 10x^3 = x^3\left(x^2-10\right)$



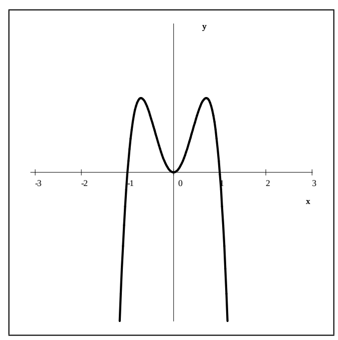
13. $f(x) = -\dfrac{1}{6}\left(x^2-9\right)\left(x^2-1\right)$
$= -\dfrac{1}{6}(x+3)(x+1)(x-1)(x-3)$



14. $f(x) = \left(x^2-25\right)\left(x^2-5\right)$
$= (x+5)\left(x+\sqrt{5}\right)\left(x-\sqrt{5}\right)(x-5)$



15. $f(x) = x^2 - x^4 = -(x+1)\,x^2\,(x-1)$



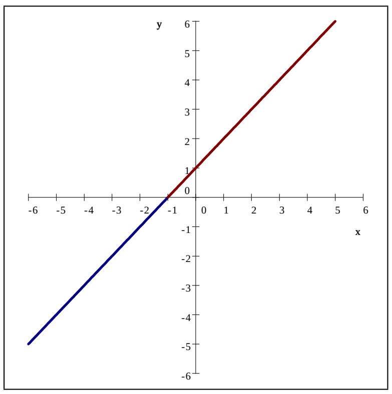
Sample Problems - Solutions

1.) Consider the expression $(x+1)\,x\,(x-3)$.
a) Find all values of x for which this expression is 0.

Solution: By the zero product rule, the equation $(x+1)\,x\,(x-3)=0$ has exactly three solutions: $x=-1$, $x=0$, and $x=3$.

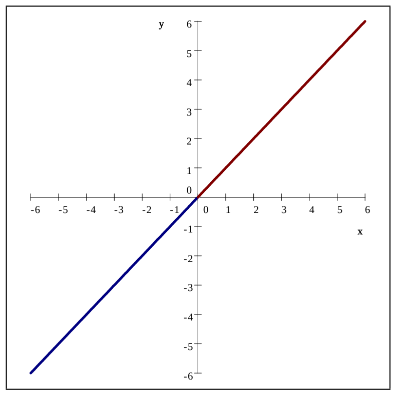
b) Find all values of x for which this expression is positive.
Solution: We will determine the sign of each linear factor and then determine the sign of the product by "counting the minus signs".

Consider the lines $y=x+1$ and $y=x$ and $y=x-3$. As the graphs below show, these linear factors have very simple behaviors. Reading their graphs left to right, they start out negative and then become zero, and then they are positive forever.



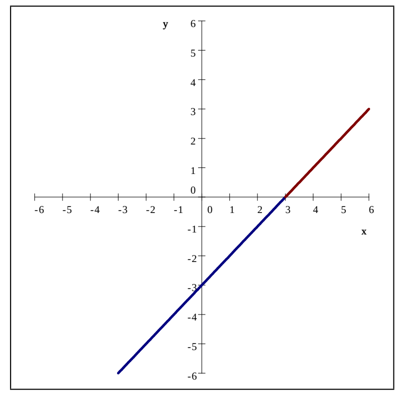
$y=x+1$

negative on $(-\infty,-1)$
zero at $x=-1$
positive on $(-1,\infty)$



$y=x$

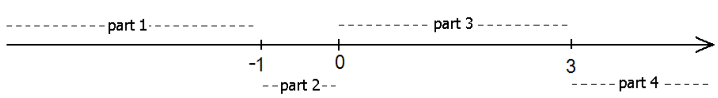
negative on $(-\infty,0)$
zero at $x=0$
positive on $(0,\infty)$



$y=x-3$

negative on $(-\infty,3)$
zero at $x=3$
positive on $(3,\infty)$

This is all the information we need to determine the sign of the expression $(x+1)\,x\,(x-3)$. We will organize this data in the table shown below. The number line will be divided into four parts by the three points $x=-1$, $x=0$, and $x=3$ as the picture below shows.



Step 1. We indicate these four intervals horizontally, and the three factors in each row as shown below. The fourth row will be for the product $(x+1)\,x\,(x-3)$.

	on $(-\infty,-1)$	on $(-1,0)$	on $(0,3)$	on $(3,\infty)$
the factor $(x+1)$				
the factor x				
the factor $x-3$				
the product $(x+1)\,x\,(x-3)$				

Step 2. We fill out the first row, indicating the sign of $x+1$ on each interval. Recall that this linear expression is negative before -1 and positive after.

	on $(-\infty,-1)$	on $(-1,0)$	on $(0,3)$	on $(3,\infty)$
the factor $(x+1)$	$-$	$+$	$+$	$+$
the factor x				
the factor $x-3$				
the product $(x+1)\,x\,(x-3)$				

We fill out the second row, indicating the sign of x on each interval. Recall that this linear expression is negative before 0 and positive after.

	on $(-\infty,-1)$	on $(-1,0)$	on $(0,3)$	on $(3,\infty)$
the factor $(x+1)$	$-$	$+$	$+$	$+$
the factor x	$-$	$-$	$+$	$+$
the factor $x-3$				
the product $(x+1)\,x\,(x-3)$				

Next, we fill out the third row, indicating the sign of $x-3$ on each interval. Recall that this linear expression is negative before 3 and positive after.

	on $(-\infty,-1)$	on $(-1,0)$	on $(0,3)$	on $(3,\infty)$
the factor $(x+1)$	$-$	$+$	$+$	$+$
the factor x	$-$	$-$	$+$	$+$
the factor $x-3$	$-$	$-$	$-$	$+$
the product $(x+1)\,x\,(x-3)$				

Step 3. We are now ready to determine the sign of te product $(x+1)\,x\,(x-3)$ on each of the intervals. For example, on the interval $(-\infty,-1)$ the product is negative becuse we are multiplying three negative numbers, as each factor is negative.

	on $(-\infty,-1)$
the factor $(x+1)$	$-$
the factor x	$-$
the factor $x-3$	$-$
the product $(x+1)\,x\,(x-3)$	$-$

On the interval $(-1,0)$ the product is positive becuse we are multiplying two negative and a positive number, as indicated by the second column. On the interval $(0,3)$, we are multiplying two positive and a negative number, so the product is negative. Finally, the product of three positive numbers is positive, and so the expression $(x+1)\,x\,(x-3)$ is positive on $(3,\infty)$. We fill out the last row accordingly and obtain the table shown below.

	on $(-\infty,-1)$	on $(-1,0)$	on $(0,3)$	on $(3,\infty)$
the factor $(x+1)$	$-$	$+$	$+$	$+$
the factor x	$-$	$-$	$+$	$+$
the factor $x-3$	$-$	$-$	$-$	$+$
the product $(x+1)\,x\,(x-3)$	$-$	$+$	$-$	$+$

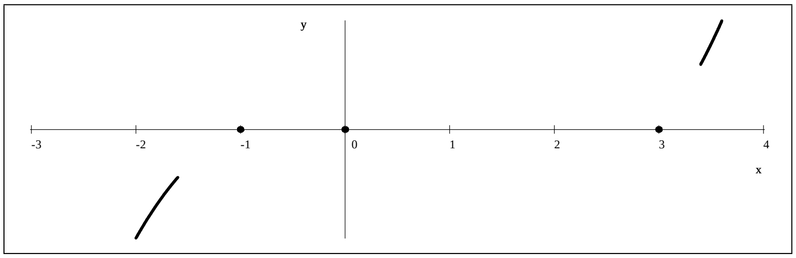
Step 4. We now read the information from the last row: the product $(x+1)\,x\,(x-3)$ is negative on $(-\infty,-1)$ and on $(0,3)$ and is positive on $(-1,0)$ and $(3,\infty)$. Thus the answer is $(-1,0)$ and $(3,\infty)$. We represent the answer using interval notation: $(-1,0)\cup(3,\infty)$.

c) Find all values of x for which this expression is negative.
Solution: Notice that the work we did for part b) also gave us the answer for part c). The answer is $(-\infty,-1)\cup(0,3)$.

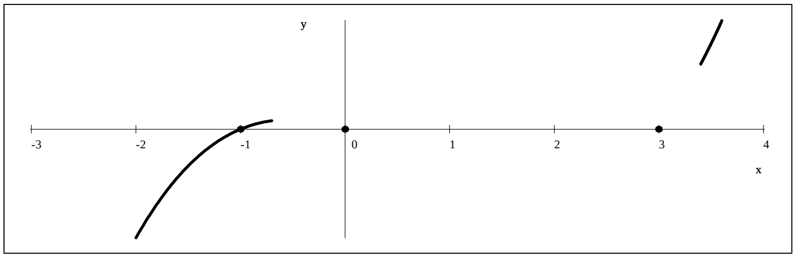
d) Based on the previous parts, sketch the graph of $f(x)=(x+1)\,x\,(x-3)$.
Solution: our graph will indicate the following facts:

1) We know that for large values of x (both positive and negative) the expression $(x+1)\,x\,(x-3)$ is large.
2) We know where the graph is positive, negative, and zero.
3) We know that all polynomials have graphs that are continuous and smooth everywhere.

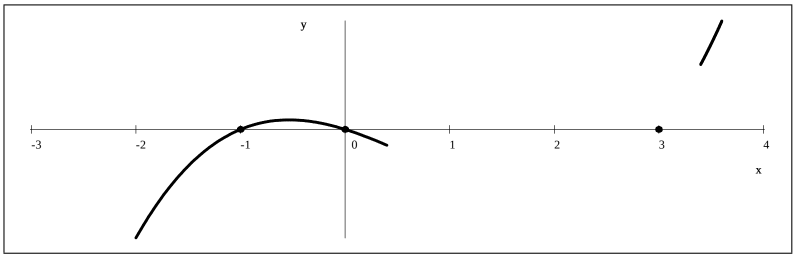
We start with the number line, indicating that f starts out large and negative, that it has exactly three zeroes, at $x=-1$, $x=0$, and $x=3$, and that it leaves the graph as large and positive.



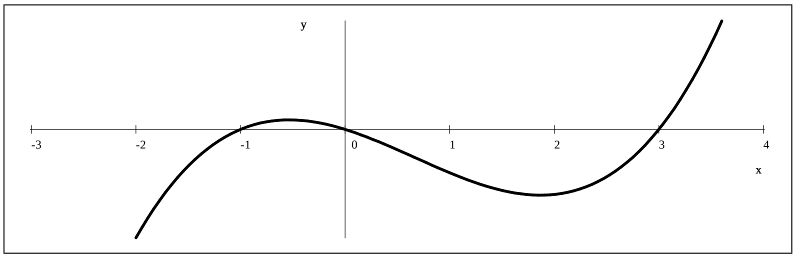
We now connect these using the following information: that polynomials are always continuous and smooth, and that we know when this polynomial is positive and negative. First, we know - from the table - that at $x=-1$, the function changes sign from negative to positive.



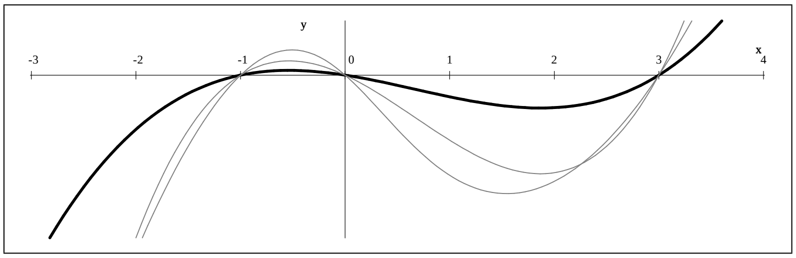
We also know that at $x=0$, the function changes sign from positive to negative.



Finally, we know that at $x=3$, the function changes sign from negative to positive.



Please note all the things we do NOT know: From our graph we see that there is a maximum somewhere between -1 and 0 and that there is a minimum somewhere between 0 and 3. We do NOT know the $x-$ or $y-$coordinate of these points. These questions will be answered in calculus. To illustrate this, consider the picture shown below. All the graphs shown are perfectly correct as they differ by things we can not determine at this level.



2.) Consider the expression $-3(x+4)(x-2)^2$.

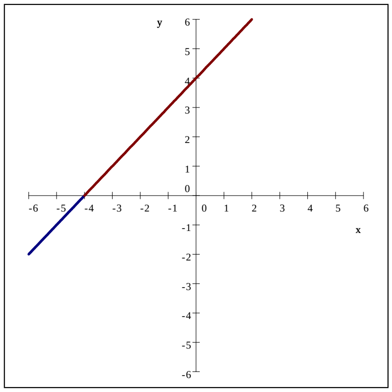
a) Find all values of x for which this expression is 0.

Solution: By the zero product rule, the equation $-3(x+4)(x-2)^2 = 0$ has exactly two solutions: $x = -4$ and $x = 2$.

b) Find all values of x for which this expression is positive.

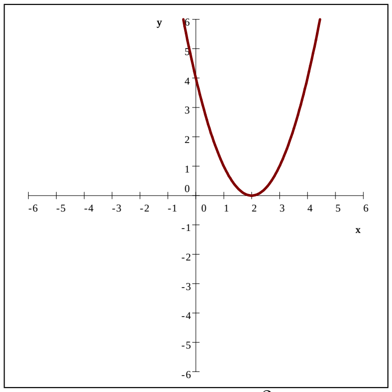
Solution: We will determine the sign of each factor and then determine the sign of the product by "counting the minus signs".

Consider the graphs $y = x + 4$ and $y = (x-2)^2$. The line $y = x + 4$ is negative before $x = -4$, is zero at $x = -4$, and then it is positive after -4. The parabola $y = (x-2)^2$ is almost always positive, except for $x = 2$, when it is zero. We will consider the coefficient -3 as a separate factor that is always zero.



$y = x + 4$

negative on $(-\infty, -4)$
zero at $x = -4$
positive on $(-4, \infty)$



$y = (x-2)^2$

positive on $(-\infty, 2)$
zero at $x = 2$
positive on $(2, \infty)$

This is all the information we need to determine the sign of the expression $-3(x+4)(x-2)^2$. We will organize this data in the table shown below. The number line will be divided into three parts by the two points $x = -4$ and $x = 2$.

We indicate these three intervals $(-\infty, -4)$, $(-4, 2)$, and $(2, \infty)$ in the top row. We indicate the three factors in each row as shown below. The fourth row will be for the product $-3(x+4)(x-2)^2$. We indicate the sign of each factor in each interval.

	on $(-\infty, -4)$	on $(-4, 2)$	on $(2, \infty)$
the factor $x+4$	$-$	$+$	$+$
the factor $(x-2)^2$	$+$	$+$	$+$
the factor -3	$-$	$-$	$-$
the product $-3(x+2)^2(x-4)$	$+$	$-$	$-$

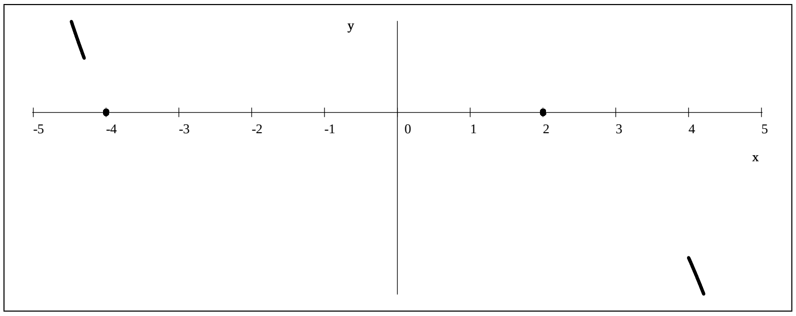
We now read the information from the last row: the product $-3(x+4)(x-2)^2$ is positive on $(-\infty, -4)$ and negative on $(-4, 2)$ and $(2, \infty)$. Thus the answer is $(-\infty, -4)$.

c) Find all values of x for which this expression is negative.

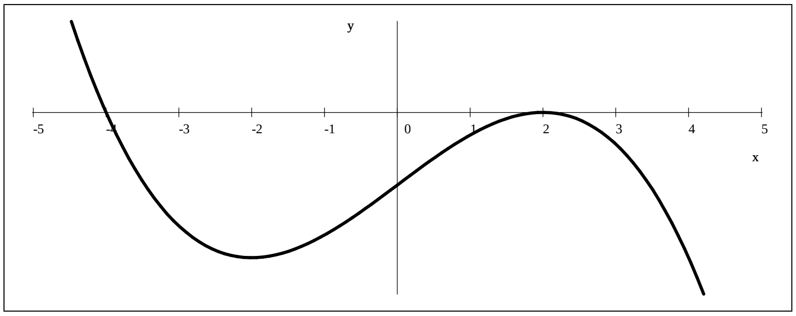
Solution: Notice that the work we did for part b) also gave us the answer for part c). The answer is $(-4, 2) \cup (2, \infty)$.

d) Based on the previous parts, sketch the graph of $f(x) = -3(x+4)(x-2)^2$.
Solution: our graph will indicate the following facts:
1) We know that for large values of x (both positive and negative) the expression $-3(x+4)(x-2)^2$ is large.
2) We know where the graph is positive, negative, and zero.
3) We know that all polynomials have graphs that are continuous and smooth everywhere.

We start with the number line, indicating that f starts out large and positive, that it has exactly two zeroes at $x = -4$ and $x = 2$, and that it leaves the graph as large and negative.



We know from the table that at $x = -4$, the function changes sign from negative to positive and that $x = 2$, the function does NOT change sign from negative to negative.



3.) Consider the expression $5x - x^3$.

a) Find all values of x for which this expression is 0.

Solution: Until now, the expressions were given in a factored form. This time we will need to factor the polynomial given. We start by the leading coefficient, -1.

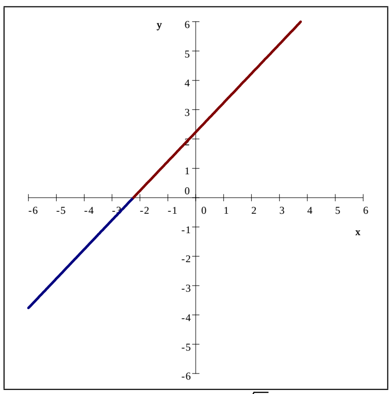
$$\begin{aligned} 5x - x^3 &= -\left(x^3 - 5x\right) && \text{factor out } x \\ &= -x\left(x^2 - 5\right) && \text{factor via the difference of squares theorem} \\ &= -x\left(x + \sqrt{5}\right)\left(x - \sqrt{5}\right) \end{aligned}$$

By the zero product rule, the equation $-x\left(x + \sqrt{5}\right)\left(x - \sqrt{5}\right) = 0$ has exactly three solutions: $x = -\sqrt{5}$, $x = 0$, and $x = \sqrt{5}$.

b) Find all values of x for which this expression is positive.

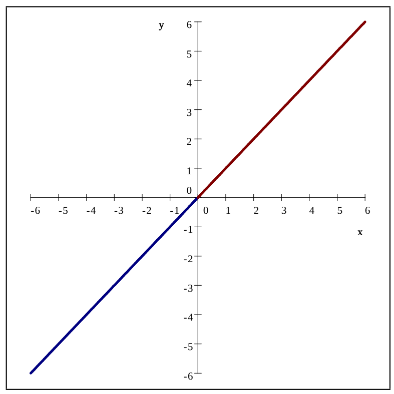
Solution: We will determine the sign of each factor and then determine the sign of the product by "counting the minus signs".

Consider the graphs of $y = x + \sqrt{5}$, $y = x$, and $y = x - \sqrt{5}$. These are all lines, negative before the zero at $-\sqrt{5}, 0$, and $\sqrt{5}$, respectively, and positive after.



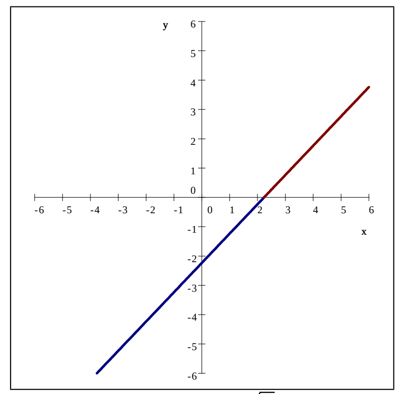
$y = x + \sqrt{5}$

negative on $\left(-\infty, -\sqrt{5}\right)$
zero at $x = -\sqrt{5}$
positive on $\left(-\sqrt{5}, \infty\right)$



$y = x$

negative on $(-\infty, 0)$
zero at $x = 0$
positive on $(0, \infty)$



$y = x - \sqrt{5}$

negative on $\left(-\infty, \sqrt{5}\right)$
zero at $x = \sqrt{5}$
positive on $\left(\sqrt{5}, \infty\right)$

This is all the information we need to determine the sign of the expression $-x\left(x + \sqrt{5}\right)\left(x - \sqrt{5}\right)$. We will organize this data in the table shown below. We will look at this product as one of four factors: x, $x + \sqrt{5}$, $x - \sqrt{5}$, and the number -1 as a separate factor that is always negative. The number line will be divided into four parts by the three points $x = -\sqrt{5}$, $x = 0$, and $x = \sqrt{5}$. We indicate the four factors in each row as shown below. The fourth row will be for the product $-x\left(x + \sqrt{5}\right)\left(x - \sqrt{5}\right)$. We indicate the sign of each factor in each interval.

	on $\left(-\infty, -\sqrt{5}\right)$	on $\left(-\sqrt{5}, 0\right)$	on $\left(0, \sqrt{5}\right)$	on $\left(\sqrt{5}, \infty\right)$
the factor $x + \sqrt{5}$	$-$	$+$	$+$	$+$
the factor x	$-$	$-$	$+$	$+$
the factor $x - \sqrt{5}$	$-$	$-$	$-$	$+$
the factor -1	$-$	$-$	$-$	$-$
the product $-x\left(x + \sqrt{5}\right)\left(x - \sqrt{5}\right)$	$+$	$-$	$+$	$-$

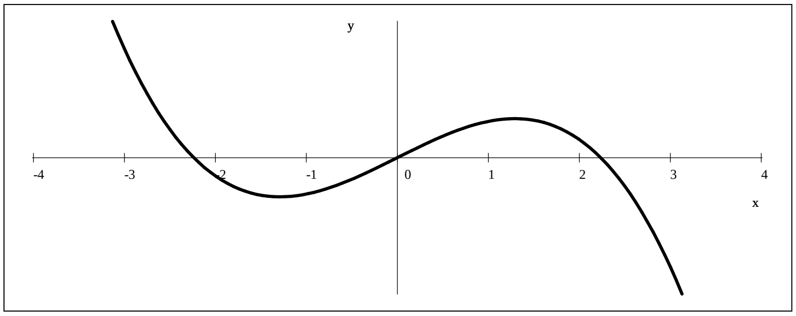
We now read the information from the last row: the product $-x\left(x+\sqrt{5}\right)\left(x-\sqrt{5}\right)$ is positive on $\left(-\infty,-\sqrt{5}\right)$ and $\left(0,\sqrt{5}\right)$, and is negative on $\left(-\sqrt{5},0\right)$ and $\left(\sqrt{5},\infty\right)$. Thus the answer is $\left(-\infty,-\sqrt{5}\right)\cup\left(0,\sqrt{5}\right)$.

c) Find all values of x for which this expression is negative.

Solution: Notice that the work we did for part b) also gave us the answer for part c). The answer is $\left(-\sqrt{5},0\right)\cup\left(\sqrt{5},\infty\right)$.

d) Based on the previous parts, sketch the graph of $f(x)=5x-x^3$.

Solution: Our graph of $f(x)=5x-x^3=-x\left(x+\sqrt{5}\right)\left(x-\sqrt{5}\right)$ will indicate the following information: that polynomials are always continuous and smooth, that for large values of x they take large values, and that we know when this polynomial is positive and negative.



4.) Consider the expression $(x+2)(x-1)^2(x-3)^3$.

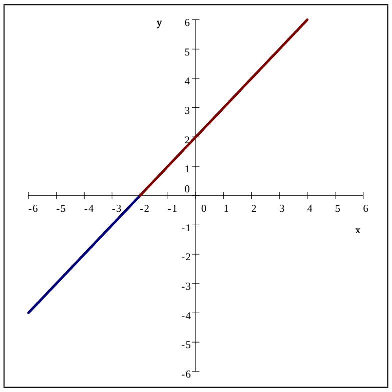
a) Find all values of x for which this expression is 0.

Solution: By the zero product rule, the equation $(x+2)(x-1)^2(x-3)^3=0$ has exactly three solutions: $x=-2$, $x=1$, and $x=3$.

b) Find all values of x for which this expression is positive.

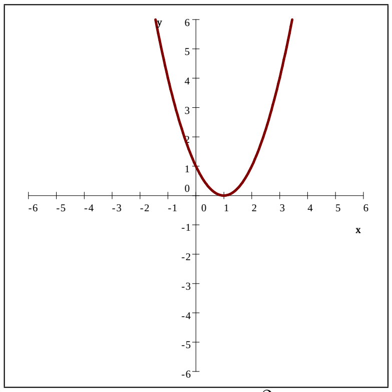
Solution: We will determine the sign of each factor and then determine the sign of the product by "counting the minus signs".

Consider the graphs of $y=x+2$, $y=(x-1)^2$, and $y=(x-3)^3$.



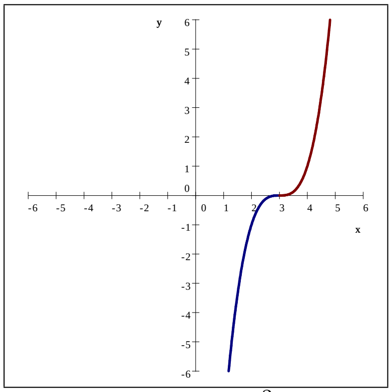
$y=x+2$

negative on $(-\infty,-2)$
zero at $x=-2$
positive on $(-2,\infty)$



$y=(x-1)^2$

positive on $(-\infty,1)$
zero at $x=1$
positive on $(1,\infty)$



$y=(x-3)^3$

negative on $(-\infty,3)$
zero at $x=3$
positive on $(3,\infty)$

This is all the information we need to determine the sign of the expression $(x+2)(x-1)^2(x-3)^3$. We will organize this data in the table shown below. We will look at this product as one of three factors: $(x+2)$, $(x-1)^2$, and $(x-3)^3$. The number line will be divided into four parts by the three points $x=-2$, $x=1$, and $x=3$. We indicate the three factors in each row as shown below. The fourth row will be for the product $(x+2)(x-1)^2(x-3)^3$. We indicate the sign of each factor in each interval.

	on $(-\infty,-2)$	on $(-2,1)$	on $(1,3)$	on $(3,\infty)$
the factor $x+2$	$-$	$+$	$+$	$+$
the factor $(x-1)^2$	$+$	$+$	$+$	$+$
the factor $(x-3)^3$	$-$	$-$	$-$	$+$
the product $(x+2)(x-1)^2(x-3)^3$	$+$	$-$	$-$	$+$

We now read the information from the last row: the product $(x+2)(x-1)^2(x-3)^3$ is positive on $(-\infty,-2)$ and $(3,\infty)$, and is negative on $(-2,1)$ and $(1,3)$. Thus the answer is $(-\infty,-2)\cup(3,\infty)$.

c) Find all values of x for which this expression is negative.

Solution: The work we did for part b) also gave us the answer for part c). The answer is $(-2,1)\cup(1,3)$.

d) Based on the previous parts, sketch the graph of $f(x)=(x+2)(x-1)^2(x-3)^3$.

Solution: We sketch the graph indicating the zeroes of the polynomial, the sign on each interval, and the fact that for large values of x, the expression is large.