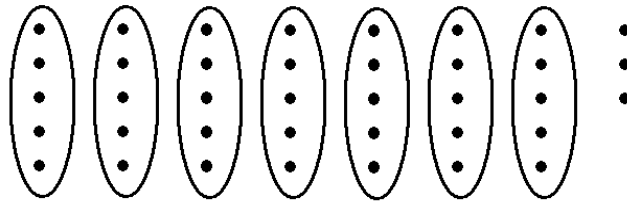


**Theorem (Division with Remainder)** For every integers  $N$  and  $m$ ,  $m \neq 0$ , there exist unique integers  $q$  and  $r$  such that

$$N = mq + r \quad \text{and} \quad r < m$$

For example, if  $N = 38$  and  $m = 5$ , then the quotient is  $q = 7$  and the remainder is  $r = 3$ . The picture below illustrates the division  $38 \div 5 = 7 \text{ R } 3$ , or, in other form:  $38 = 5 \cdot 7 + 3$ .



Note that there are two different ways to represent this division:  $38 \div 5 = 7 \text{ R } 3$  and  $\frac{38}{5} = 7\frac{3}{5}$

This theorem, also called the Euclidean division, is very fundamental to mathematics. There is a similar theorem about division of polynomials.

**Theorem (Division with Remainder for Polynomials)** For every polynomials  $N(x)$  and  $m(x)$ ,  $m(x)$  not the constant zero polynomial, there exist unique polynomials  $q(x)$  and  $r(x)$  such that

$$N(x) = m(x)q(x) + r(x) \quad \text{and the degree of } r(x) \text{ is less than that of } m(x)$$

A special case of this theorem is when we divide by a linear polynomial. By the theorem above, such a division results in a remainder that is a polynomial with a degree smaller than linear: that is, a constant polynomial.

Suppose that  $N(x)$  is a polynomial and  $m(x) = x - a$  is a linear polynomial. According to the theorem above, there exist a polynomial  $q(x)$  and real number  $r$  such that

$$N(x) = q(x)(x - a) + r$$

The equality above is true between polynomial functions. Thus, the equality above is true for all values of  $x$ . Let  $x = a$ .

$$N(x) = q(x)(x - a) + r \quad \text{Let } a = x$$

$$N(a) = q(a)(a - a) + r$$

$$N(a) = q(a) \cdot 0 + r$$

$$N(a) = r$$

**Theorem (Remainder Theorem)** Suppose that  $N(x)$  is a polynomial and  $m(x) = x - a$ . The remainder of the division of  $N(x)$  by  $x - a$  is equal to  $N(a)$ .

An immediate consequence of this theorem is when this remainder is zero.

**Theorem:** Suppose that  $N(x)$  is a polynomial and  $a$  is a real number such that  $N(a) = 0$ . Then  $x - a$  is a factor of  $N(x)$ , i.e. there exists a polynomial  $q(x)$  such that  $N(x) = (x - a)q(x)$



## Sample Problems

1. Use the remainder theorem to find the remainder in the division  $(5x^4 - 8x^3 + 3x^2 - x - 1) \div (x - 2)$
2. Solve the equation  $x^7 - 17x^5 + 36x^4 - 20x^3 = 0$  given that 2 is a solution of the equation.
3. Solve the equation  $x^6 - 2x^5 - 50x^4 + 4x^3 + 97x^2 - 2x - 48 = 0$ .



## Practice Problems

1. Perform the indicated divisions with remainders:
  - a)  $132 \div 7$
  - b)  $1145 \div 12$
  - c)  $918 \div 8$
  - d)  $201 \div 12$
2. Use the remainder theorem to find the remainder in each of the following divisions.
  - a)  $(8x^5 - 2x^3 + x + 11) \div (x - 1)$
  - b)  $(8x^5 - 2x^3 + x + 11) \div (x + 1)$
  - c)  $(x^5 + 3x^4 + 2x^3 - x^2 - 3x - 2) \div (x - 2)$
  - d)  $(x^5 + 3x^4 + 2x^3 - x^2 - 3x - 2) \div (x + 2)$
3. Solve each of the given equations.
  - a)  $x^4 + 7x^3 - 4x^2 - 28x = 0$  given that  $-2$  is a solution
  - b)  $x^6 + x^5 - 18x^4 - 52x^3 - 40x^2 = 0$  given that  $2$  is a solution
  - c)  $x^4 + 8x^3 + 12x^2 - 32x - 64 = 0$  given that  $-4$  is a solution
4. Solve each of the following equations.
  - a)  $x^6 - 15x^5 + 53x^4 - 21x^3 - 90x^2 = 0$
  - b)  $x^6 + 7x^5 - 47x^4 - 307x^3 - 394x^2 + 236x + 504 = 0$
5. Find all the values of  $k$  such that  $f(x) = k^2x^3 - 7kx + 12$  is divisible by  $(x - 1)$ .



## Answers

## Sample Problems

1. 25    2.  $-5, 0, 1, 2$     3.  $-6, -1, 1, 8$

## Practice Problems

1. a) 18 R 6    b) 95 R 5    c) 114 R 6    d) 16 R 9    2. a) 18    b) 4    c) 84    d) 0  
3. a)  $-7, -2, 0, 2$     b)  $-2, 0, 5$     c)  $-4, -2, 2$     4. a)  $-1, 0, 3, 10$     b)  $-9, -2, 1, 7$     5. 3, 4

## Solutions



## Sample Problems

1. Use the remainder theorem to find the remainder in the division  $(5x^4 - 8x^3 + 3x^2 - x - 1) \div (x - 2)$

Solution: Let  $N(x) = 5x^4 - 8x^3 + 3x^2 - x - 1$ . If we divide this polynomial by  $m(x) = x - 2$ , the remainder is the same as  $N(a)$ . Thus, we evaluate  $N(x)$  at  $x = 2$ .

$$\begin{aligned} N(2) &= 5 \cdot 2^4 - 8 \cdot 2^3 + 3 \cdot 2^2 - 2 - 1 = 5 \cdot 16 - 8 \cdot 8 + 3 \cdot 4 - 2 - 1 \\ &= 80 - 64 + 12 - 2 - 1 = 25 \end{aligned}$$

Thus the remainder is 25.

- b)  $(x^5 + 3x^4 - x^3 - 2x^2 + x - 6) \div (x + 3)$

Solution: Let  $N(x) = x^5 + 3x^4 - x^3 - 2x^2 + x - 6$ . If we divide this polynomial by  $m(x) = x + 3$ , the remainder is the same as  $N(-3)$ . Thus, we evaluate  $N(x)$  at  $x = -3$ .

$$\begin{aligned} N(2) &= x^5 + 3x^4 - x^3 - 2x^2 + x - 6 = (-3)^5 + 3(-3)^4 - (-3)^3 - 2(-3)^2 + (-3) - 6 \\ &= -243 + 3 \cdot 81 - (-27) - 2 \cdot 9 - 3 - 6 = -243 + 243 + 27 - 18 - 3 - 6 = 0 \end{aligned}$$

Thus the remainder is 0. This means that  $(x + 3)$  is a linear factor of  $N(x)$ . Indeed, if we perform the division, the quotient is  $x^4 - x^2 + x - 2$  and the remainder is 0.

2. Solve the equation  $x^7 - 17x^5 + 36x^4 - 20x^3 = 0$  given that 2 is a solution of the equation.

Solution: To solve any equation of degree greater than one, we need to factor and apply the zero product rule. The greatest common factor is  $x^3$ . We factor it out:

$$x^7 - 17x^5 + 36x^4 - 20x^3 = x^3(x^4 - 17x^2 + 36x - 20)$$

We still need to factor  $P(x) = x^4 - 17x^2 + 36x - 20$ . Since  $P(2) = 0$ ,  $x - 2$  is a linear factor of  $P(x)$ . We divide  $x^4 - 17x^2 + 36x - 20$  by  $x - 2$ . (For the steps to dividing a polynomials, please see the separate handout on dividing polynomials.) The quotient is  $x^3 + 2x^2 - 13x + 10$ . So far, we have that

$$x^7 - 17x^5 + 36x^4 - 20x^3 = x^3(x - 2)(x^3 + 2x^2 - 13x + 10)$$

We still need to factor  $Q(x) = x^3 + 2x^2 - 13x + 10$ . Let us substitute  $x = 2$  into  $Q(x)$ .

$$Q(2) = 2^3 + 2 \cdot 2^2 - 13 \cdot 2 + 10 = 8 + 8 - 26 + 10 = 0$$

This means that  $x - 2$  is a factor of  $Q(x)$ . We divide:  $x^3 + 2x^2 - 13x + 10$  by  $x - 2$  and obtain the quotient  $x^2 + 4x - 5$  and the remainder is zero. So now we have

$$x^7 - 17x^5 + 36x^4 - 20x^3 = x^3(x - 2)(x - 2)(x^2 + 4x - 5) = x^3(x - 2)^2(x^2 + 4x - 5)$$

We factor the quadratic expression  $x^2 + 4x - 5$  and obtain  $(x + 5)(x - 1)$ . Thus, we now have completely factored our polynomial:

$$x^7 - 17x^5 + 36x^4 - 20x^3 = 0$$

$$x^3(x - 2)^2(x + 5)(x - 1) = 0$$

$$x_1 = 0, x_2 = 2, x_3 = -5, x_4 = 1$$

3. Solve the equation  $x^6 - 2x^5 - 50x^4 + 4x^3 + 97x^2 - 2x - 48 = 0$ .

Solution: Since this equation is of degree 6, we do not have obvious factoring techniques. So, we will rely on the remainder theorem. Let  $P(x) = x^6 - 2x^5 - 50x^4 + 4x^3 + 97x^2 - 2x - 48$ . In order to find some factors, we will be looking for zeroes of the polynomial. We start by substituting easy, small numbers such as 1, -1, 2, and -2.  $x = 1$  turns out to be a zero. So,  $x - 1$  is a factor. We divide:

$$(x^6 - 2x^5 - 50x^4 + 4x^3 + 97x^2 - 2x - 48) \div (x - 1) = x^5 - x^4 - 51x^3 - 47x^2 + 50x + 48$$

So we have

$$P(x) = (x^5 - x^4 - 51x^3 - 47x^2 + 50x + 48)(x - 1)$$

We substitute  $x = 1$  into the quotient  $x^5 - x^4 - 51x^3 - 47x^2 + 50x + 48$  and obtain zero. Thus,  $x - 1$  is still a factor. We divide

$$(x^5 - x^4 - 51x^3 - 47x^2 + 50x + 48) \div (x - 1) = x^4 - 51x^2 - 98x - 48$$

So now we have that

$$P(x) = (x^4 - 51x^2 - 98x - 48)(x - 1)^2$$

We substitute  $x = 1$  into the quotient  $x^4 - 51x^2 - 98x - 48$  and obtain  $-196$ . This means that we have exhausted  $x = 1$ .

Let us try  $x = -1$ . We evaluate  $Q(x) = x^4 - 51x^2 - 98x - 48$  at  $x = -1$ . Since the result is zero,  $x + 1$  is a factor of  $Q(x)$ . We divide

$$(x^4 - 51x^2 - 98x - 48) \div (x + 1) = x^3 - x^2 - 50x - 48$$

So we now have that

$$P(x) = (x^3 - x^2 - 50x - 48)(x - 1)^2(x + 1)$$

We evaluate  $T(x) = x^3 - x^2 - 50x - 48$  at  $x = -1$ . Since the result is zero,  $x + 1$  is a factor of  $T(x)$ . We divide

$$(x^3 - x^2 - 50x - 48) \div (x + 1) = x^2 - 2x - 48$$

So we now have that

$$P(x) = (x^2 - 2x - 48)(x - 1)^2(x + 1)^2$$

We can easily factor the quadratic factor:  $x^2 - 2x - 48 = (x + 6)(x - 8)$ . Now we have completely factored our polynomial.

$$x^6 - 2x^5 - 50x^4 + 4x^3 + 97x^2 - 2x - 48 = 0$$

$$(x + 6)(x - 8)(x - 1)^2(x + 1)^2 = 0$$

$$x_1 = -6, x_2 = 8, x_3 = 1, x_4 = -1$$

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