Definition: A rational function is a function that can be written as a quotient of two polynomials. For example, $f(x) = \dfrac{1}{x}$ and $h(x) = \dfrac{x^2-2}{3x+1}$ are rational functions.

While polynomials are continuous on $\mathbb{R}$ (the set of all real numbers), rational functions might be discontinuous (i.e. not continuous) at some points. For example, $f(x) = \dfrac{1}{x}$ is discontinuous at $x = 0$.

Theorem: Suppose that $P(x)$ and $Q(x)$ are polynomials. Then $r(x) = \dfrac{P(x)}{Q(x)}$ is continuous everywhere, except for the zeroes of $Q(x)$.

It is very important for us to remember that all discontinuities of rational functions are at the zeroes of its denominator. For example, $t(x) = \dfrac{1}{x^2+1}$ is a rational function that is continuous everywhere because its denominator is never zero.

Discontinuities of functions can look in various ways. In case of rational functions, there are only two types of discontinuities, and they are both related to division by zero.

Consider the rational functions $f(x) = \dfrac{1}{x}$ and $g(x) = \dfrac{x}{x}$. We will see some significant similarities and differences.

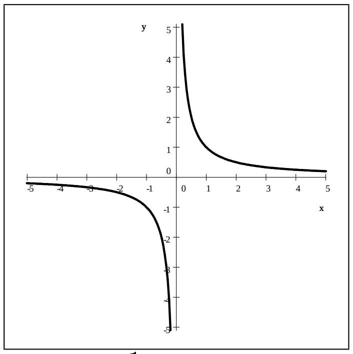
Similarities: the two functions have the same domain, all real numbers, except for zero. At $x = 0$, we would be dividing by zero in case of both functions. However, the differences are about what happens **near zero**.

We already understand that in case of $f(x) = \dfrac{1}{x}$. There, we are taking the reciprocal of extremely small numbers - those are extremely large numbers. So, the closer we get to zero on the x-axis, the larger the $y-$values become. This creates a discontinuity we describe as a vertical asymptote.

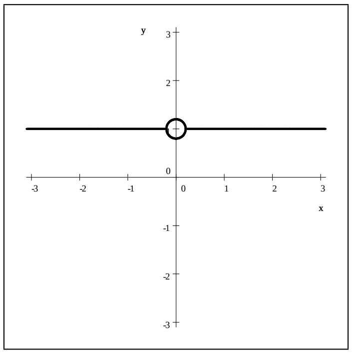
This is not what happens in the case of $g(x) = \dfrac{x}{x}$. The equation for this function can be simplified

$$g(x) = \begin{cases} 1 & \text{if } x \neq 0 \\ \text{undefined} & \text{if } x = 0 \end{cases}$$

In case of $g(x)$, the extremely small (but non-zero) value of x appears both in numerator and denominator and is cancelled out. Near zero, this function has a constant value of 1. **So in this case, the zero in the denominator, is only causing strange behavior at zero, but not near it.** Such a discontinuity is called a hole or a removable discontinuity.



$f(x) = \dfrac{1}{x}$ has a vertical asymptote at $x = 0$



$g(x) = \dfrac{x}{x}$ has a hole at $x = 0$

Both holes and vertical asymptotes are caused by a zero of the denominator. But in case of a hole, the zero appears both in the numerator and denominator, cancelling each other everywhere except at the zero itself.

Theorem: Suppose that $P(x)$ and $Q(x)$ are polynomials. All discontinuities of $r(x) = \dfrac{P(x)}{Q(x)}$ are at the zeroes of $Q(x)$. These discontinuities are of the following two types:

1) Holes: $r(x) = \dfrac{P(x)}{Q(x)}$ has a hole at $x = a$ if a is a zero of the denominator and the numerator, and after cancellation of common $(x-a)$ factors from numerator and denominator, $(x-a)$ is not a factor of the denominator. In other words, **holes are zeroes of the denominator before cancellation but not after**.

2) Vertical Asymptotes: $r(x) = \dfrac{P(x)}{Q(x)}$ has a vertical asymptote at $x = a$ if a is a zero of the denominator even after cancellation of common $(x-a)$ factors from numerator and denominator. In short, **vertical asymptotes are zeroes of the denominator after cancellation**.

The distinction can be easily articulated through notation. Consider the function

$$f(x) = \frac{-2x(x+3)(x+1)}{3(x+3)(x+1)^2}$$

We can simplify this function by cancelling out common factors from numerator and denominator. However, we must do so carefully as to not to change the domain of the function. Recall that two functions are equal if they have the same assignment AND the same domain.

Consider the factor $(x+3)$. We can cancel it out, but that would change the domain, as $x = -3$ would become part of the domain. In other words, $\dfrac{x+3}{x+3} = 1$ is not a true equality between functions. What is true is that

$$\frac{x+3}{x+3} = \begin{cases} 1 & \text{if } x \neq -3 \\ \text{undefined} & \text{if } x = -3 \end{cases}$$

We need to make sure that we do not change the domain when simplifying the equation for the function.

$$f(x) = \frac{-2x(x+3)(x+1)}{3(x+3)(x+1)^2} = \begin{cases} \dfrac{-2x(x+1)}{3(x+1)^2} & \text{if } x \neq -3 \\ \text{undefined} & \text{if } x = -3 \end{cases}$$

Consider now the factor $(x+1)$. We can cancel it out, and the domain would remain the same because $\dfrac{x+1}{(x+1)^2}$ and $\dfrac{1}{x+1}$ have the same domain: all real numbers except for $x = -1$. Since $(x+1)$ is still a zero of the denominator after cancellation, we do not have to treat it as $x = -3$.

$$f(x) = \frac{-2x(x+3)(x+1)}{3(x+3)(x+1)^2} = \begin{cases} \dfrac{-2x}{3(x+1)} & \text{if } x \neq -3 \\ \text{undefined} & \text{if } x = -3 \end{cases}$$

As it often happens in mathematics, both forms of the function's equation are useful. The un-simplified version offers easy access to the list of all discontinuities: at $x = -3$ and -1. The carefully simplified form offers an easy way to decide what types are the discontinuities: the function has a vertical asymptote at $x = -1$ because it is a zero of the denominator even after the cancellation. The function has a hole at $x = -3$ because it is a zero of the denominator before cancellation but not after. Indeed, we needed to artificially remind ourselves that $x = -3$ is not in the domain.

Practice Problems

In each of the following functions given,

a) list all discontinuities
b) simplify the function's equation
c) determine which discontinuities are holes and which are vertical asymptotes.

1. $f(x) = \dfrac{(x+2)^3 x^2 (x-2)^6 (x-5)^2}{(x+5)(x+2)x^3(x-2)^4(x-5)^4}$

2. $f(x) = \dfrac{(x+7)(x+4)^3 x^2 (x-2)^5 (x-3)^3}{(x+4)(x+1)x^2(x-2)^4(x-3)^4}$

3. $f(x) = \dfrac{(x+2)^4 x (x-2)^6 (x-5)^2}{(x+3)x^3(x-2)(x-5)^4}$

4. $f(x) = \dfrac{(x+3)(x+1)x^2(x-2)(x-5)^3}{(x+3)^4(x+1)x(x-2)^2(x-5)^2(x-7)}$

Answers - Practice Problems

1. $f(x) = \dfrac{(x+2)^3 x^2 (x-2)^6 (x-5)^2}{(x+5)(x+2)x^3(x-2)^4(x-5)^4}$

   a) discontinuities: $x = -5, -2, 0, 2, 5$

   b) $f(x) = \begin{cases} \dfrac{(x+2)^2 (x-2)^2}{(x+5)\,x\,(x-5)^2} & \text{if } x \neq -2, 2 \\ \text{undefined} & \text{if } x = -2, 2 \end{cases}$

   c) vertical asymptotes at $x = -5, 0, 5$ holes at $x = -2, 2$

2. $f(x) = \dfrac{(x+7)(x+4)^3 x^2 (x-2)^5 (x-3)^3}{(x+4)(x+1)x^2(x-2)^4(x-3)^4}$

   a) discontinuities: $x = -4, -1, 0, 2, 3$

   b) $f(x) = \begin{cases} \dfrac{(x+7)(x+4)^2(x-2)}{(x+1)(x-3)} & \text{if } x \neq -4, 0, 2 \\ \text{undefined} & \text{if } x = -4, 0, 2 \end{cases}$

   c) vertical asymptotes at $x = -1, 3$ holes at $x = -4, 0, 2$

3. $f(x) = \dfrac{(x+2)^4 x (x-2)^6 (x-5)^2}{(x+3)x^3(x-2)(x-5)^4}$

   a) discontinuities: $x = -3, 0, 2, 5$

   b) $f(x) = \begin{cases} \dfrac{(x+2)^4 (x-2)^5}{(x+3)\,x^2\,(x-5)^2} & \text{if } x \neq 2 \\ \text{undefined} & \text{if } x = 2 \end{cases}$

   c) vertical asymptotes at $x = -3, 0, 5$ holes at $x = 2$

4. $f(x) = \dfrac{(x+3)(x+1)x^2(x-2)(x-5)^3}{(x+3)^4(x+1)x(x-2)^2(x-5)^2(x-7)}$

   a) discontinuities: $x = -3, -1, 0, 2, 5, 7$

   b) $f(x) = \begin{cases} \dfrac{x(x-5)}{(x+3)^3(x-2)(x-7)} & \text{if } x \neq -1, 0, 5 \\ \text{undefined} & \text{if } x = -1, 0, 5 \end{cases}$

   c) vertical asymptotes at $x = -3, 2, 7$ holes at $x = -1, 0, 5$

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