Sample Problems

Recall the following identities. For all natural numbers n,

$$1+2+3+...+n = \frac{n(n+1)}{2}$$
$$1^2+2^2+3^2+...+n^2 = \frac{n(n+1)(2n+1)}{6}$$
$$1^3+2^3+3^3+...+n^3 = \frac{n^2(n+1)^2}{4}$$

Compute each of the following sums.

1. $\displaystyle\sum_{k=0}^{20}\left(k^2+2\right)$
2. $\displaystyle\sum_{k=0}^{100}\left(k^3-k^2-2k+1\right)$
3. $0.5^2+1^2+1.5^2+2^2+2.5^2+...+39.5^2+40^2$
4. a) $\displaystyle\sum_{m=1}^{100}(3m)$ b) $\displaystyle\sum_{m=1}^{100}5m^2$
5. a) $2^2+4^2+6^2+....+100^2$
   b) $1^2+3^2+5^2+7^2+....+99^2$
6. $100^2+101^2+102^2+....+400^2$

Practice Problems

Compute each of the following sums.

1. $\displaystyle\sum_{n=0}^{30}\left(n^2-1\right)$
2. $\displaystyle\sum_{n=0}^{28}\left(n^2-6n+9\right)$
3. $\displaystyle\sum_{n=1}^{100}\left(n^2-(n-1)^2\right)$
4. $\displaystyle\sum_{k=1}^{20}\left(k^3+2k\right)$
5. $\displaystyle\sum_{m=1}^{99}\left(m^2-2m+5\right)$
6. $\displaystyle\sum_{j=0}^{50}\left(2j^2\right)$
7. $\displaystyle\sum_{j=0}^{50}(2j)^2$
8. $5^2+10^2+15^2+...+100^2$
9. $0.2^2+0.4^2+0.6^2+....+19.8^2+20^2$
10. $0.5^2+1^2+1.5^2+...+34.5^2+35^2$
11. $\left(\frac{1}{6}\right)^2+\left(\frac{2}{6}\right)^2+\left(\frac{3}{6}\right)^2+....+\left(\frac{120}{6}\right)^2$
12. $10^3+11^3+12^3+...+30^3$
13. $\displaystyle\sum_{t=50}^{100}\left(t^2-t\right)$
14. $1^3+3^3+5^3+...+99^3$

Sample Problems - Answers

1. 2912
2. 25 154 151
3. 43 470
4. a) 15 150 b) 1691 750
5. a) 171 700 b) 166 650
6. 21 085 050

Practice Problems - Answers

1. 9424
2. 5539
3. 10 000
4. 44 520
5. 318 945
6. 85 850
7. 171 700
8. 71 750
9. 13 534
10. 29198. 75
11. $\dfrac{145\,805}{9} = 16200.\overline{5}$
12. 214 200
13. 294 100
14. 12 497 500

Sample Problems - Solutions

Recall the following identities. For all natural numbers n,

$$\begin{aligned} 1+2+3+\ldots+n &= \frac{n(n+1)}{2} \\ 1^2+2^2+3^2+\ldots+n^2 &= \frac{n(n+1)(2n+1)}{6} \\ 1^3+2^3+3^3+\ldots+n^3 &= \frac{n^2(n+1)^2}{4} \end{aligned}$$

Compute each of the following sums.

1. $\displaystyle\sum_{k=0}^{20}\left(k^2+2\right)$

Solution: We separate the constant part from the variable part and sum them separately. One part that is tricky is to realize that we are adding 2 twenty-one times (and not twenty times).

$$\begin{aligned} \sum_{k=0}^{20}\left(k^2+2\right) &= \left(0^2+2\right)+\left(1^2+2\right)+\left(2^2+2\right)+\ldots+\left(20^2+2\right) \\ &= \left(0^2+1^2+2^2+\ldots+20^2\right)+(2+2+2+\ldots+2) \\ &= \left(1^2+2^2+\ldots+20^2\right)+21\cdot 2 = \frac{20(20+1)(2\cdot 20+1)}{6}+42 \\ &= \frac{20\cdot 21\cdot 41}{6}+42 = 2870+42 = \boxed{2912} \end{aligned}$$

2. $\displaystyle\sum_{k=0}^{100}\left(k^3-k^2-2k+1\right)$

Solution: This problem can be made easier by a smart presentation of the terms to be added, in a rectangular form

$$\begin{array}{lccccccccc} k=0 & 0^3 & - & 0^2 & - & 2\cdot 0 & + & 1 \\ k=1 & 1^3 & - & 1^2 & - & 2\cdot 1 & + & 1 \\ k=2 & 2^3 & - & 2^2 & - & 2\cdot 2 & + & 1 \\ k=3 & 3^3 & - & 3^2 & - & 2\cdot 3 & + & 1 \\ \vdots & \vdots & & \vdots & & \vdots & & \vdots \\ k=100 & 100^3 & - & 100^2 & - & 2\cdot 100 & + & 1 \end{array}$$

We will deal with each column separately. The first column is

$$S_1=0^3+1^3+2^3+3^3+...+100^3=1^3+2^3+3^3+...+100^3=\left(\frac{100\cdot 101}{2}\right)^2=25\,502\,500$$

The second column is

$$S_2=-0^2-1^2-2^2-3^2-...-100^2=-\left(1^2+2^2+3^2+...+100^2\right)=-\frac{100\cdot 101\cdot 201}{6}=-338\,350$$

The third column is

$$S_3=-2\cdot 0-2\cdot 1-2\cdot 2-2\cdot 3-....-2\cdot 100=-2\left(1+2+3+...+100\right)=-2\ \frac{100\cdot 101}{2}=-10\,100$$

And the fourth column is (again, we must be careful with the $k=0$ term that does contribute 1)

$$S_4=1+1+1+...+1=101$$

So the sum is then

$$\sum_{k=0}^{100}\left(k^3-k^2-2k+1\right)=25\,502\,500-338\,350-10\,100+101=\boxed{25\,154\,151}$$

3. $0.5^2+1^2+1.5^2+2^2+2.5^2+...+39.5^2+40^2$

Solution: In case of this problem, fractions are much more helpful than decimals. Let us express this sum in terms of fractions first. Then we will bring every term to the same denominator to better see the patterns.

$$\begin{aligned} S &= 0.5^2+1^2+1.5^2+2^2+2.5^2+......+39.5^2+40^2 \\ &= \left(\frac{1}{2}\right)^2+1^2+\left(\frac{3}{2}\right)^2+2^2+\left(\frac{5}{2}\right)^2+......+\left(\frac{79}{2}\right)^2+40^2 \\ &= \left(\frac{1}{2}\right)^2+\left(\frac{2}{2}\right)^2+\left(\frac{3}{2}\right)^2+\left(\frac{4}{2}\right)^2+\left(\frac{5}{2}\right)^2+......+\left(\frac{79}{2}\right)^2+\left(\frac{80}{2}\right)^2 \\ &= \frac{1^2}{4}+\frac{2^2}{4}+\frac{3^2}{4}+\frac{4^2}{4}+\frac{5^2}{4}+...+\frac{79^2}{4}+\frac{80^2}{4}=\frac{1^2+2^2+3^2+...+80^2}{4}=\frac{1}{4}\cdot\frac{80\cdot 81\cdot 161}{6}=\boxed{43\,470} \end{aligned}$$

4. a) $\displaystyle\sum_{m=1}^{100}(3m)$

Solution:

$$\sum_{m=1}^{100}(3m)=3+6+9+12+....+300=3(1+2+3+4+...+100)=3\cdot\frac{100\cdot 101}{2}=\boxed{15\,150}$$

b) $\displaystyle\sum_{m=1}^{100}5m^2$

Solution:

$$\sum_{m=1}^{100}\left(5m^2\right)=5\cdot 1^2+5\cdot 2^2+5\cdot 3^2+....+5\cdot 100^2=5\left(1^2+2^2+3^2+...+100^2\right)=5\cdot\frac{100\cdot 101\cdot 201}{6}=\boxed{1691\,750}$$

5. a) $2^2+4^2+6^2+....+100^2$

Solution:

$$\begin{aligned}S &= 2^2+4^2+6^2+...+100^2=(2\cdot 1)^2+(2\cdot 2)^2+(2\cdot 3)^2+...+(2\cdot 50)^2=\\ &= 4\cdot 1^2+4\cdot 2^2+4\cdot 3^2+...+4\cdot 50^2=4\left(1^2+2^2+3^2+...+50^2\right)=4\cdot\frac{50\cdot 51\cdot 101}{6}=\boxed{171\,700}\end{aligned}$$

b) $1^2+3^2+5^2+7^2+....+99^2$

Solution:

$$\begin{aligned}S &= \left(1^2+2^2+3^2+...+100^2\right)-\left(2^2+4^2+6^2+...+100^2\right)\\ &= \frac{100\cdot 101\cdot 201}{6}-\left((2\cdot 1)^2+(2\cdot 2)^2+(2\cdot 3)^2+...+(2\cdot 50)^2\right)\\ &= 338\,350-4\left(1^2+2^2+3^2+...+50^2\right)=338\,350-171\,700=\boxed{166\,650}\end{aligned}$$

6. $100^2+101^2+102^2+....+400^2$

Solution:

$$\begin{aligned}S &= 100^2+101^2+102^2+....+400^2=\left(1^2+2^2+3^2+....+400^2\right)-\left(1^2+2^2+2^2+....+99^2\right)\\ &= \frac{400\cdot 401\cdot 801}{6}-\frac{99\cdot 100\cdot 199}{6}=\boxed{21\,085\,050}\end{aligned}$$

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