Translated from the work of Zoltan Tuzson, teacher in Szekelyudvarhely.

Notation: a, b, and c will denote the sides of a triangle, A the area, P the perimeter, R the radius of circumscribed circle, r the radius of the inscribed circle, α , β , γ the angles opposite to side a, b, and c, correspondingly, h_a, h_b, and h_c the height belonging to side a, b, and c, correspondingly. Let m_a, m_b, and m_c denote the medians belonging to each side. (A median is the line segment connecting the midpoint of a side with its opposite vertex.) Let i_α, i_β, and i_γ denote the length of the angle bisectors (the line segment that intersects the triangle).

Prove each of the following.

1. In any triangle, $A = \dfrac{1}{2}ab\sin\gamma$.
2. (the law of sines) In any triangle, $\dfrac{a}{\sin\alpha} = \dfrac{b}{\sin\beta} = \dfrac{c}{\sin\gamma} = 2R$.
3. In any triangle, $R = \dfrac{abc}{4A}$.
4. In any triangle, $a = b\cos\gamma + c\cos\beta$.
5. In any triangle, $b\cos\beta + c\cos\gamma = a\cos(\beta - \gamma)$.
6. (the law of cosines) In any triangle, $c^2 = a^2 + b^2 - 2ab\cos\gamma$.
7. In any triangle, $b\cos\gamma - c\cos\beta = \dfrac{b^2 - c^2}{a}$.
8. In any triangle, $A = \dfrac{a^2\sin\beta\sin\gamma}{2\sin\alpha}$.
9. If $a\sin\beta + b\sin\alpha = 2c\sin\gamma$ and $b\sin\gamma + c\sin\beta = 2a\sin\alpha$, then the triangle is equilateral.
10. In any triangle, $\sqrt{a\sin\alpha} + \sqrt{b\sin\beta} + \sqrt{c\sin\gamma} = \sqrt{P(\sin\alpha + \sin\beta + \sin\gamma)}$.
11. In any triangle, $\sin\alpha \cdot \sin\beta \cdot \sin\gamma = \dfrac{A}{2R^2}$.
12. In any triangle, $\sin\alpha + \sin\beta + \sin\gamma = \dfrac{P}{2R}$.
13. If $b + c = 3a$, then $\sin\beta + \sin\gamma = 3\sin\alpha$.
14. If $b\cos\gamma = c\cos\beta$, then the triangle is isosceles.
15. If $b^2 + 2ac\cos\beta = a^2 + 2bc\cos\alpha$, then the triangle is isosceles.
16. If $a = 2b\cos\gamma$, then the triangle is isosceles.
17. If $a\cos\alpha = b\cos\beta$, then the triangle is isosceles or has a right angle.
18. If $\cos\beta + \cos\gamma = \dfrac{b+c}{a}$, then the triangle has a right angle.
19. In any triangle, $\dfrac{\cos\alpha}{a} + \dfrac{\cos\beta}{b} + \dfrac{\cos\gamma}{c} = \dfrac{a^2 + b^2 + c^2}{2abc}$.
20. In any triangle, $(a+b)\cos\gamma + (b+c)\cos\alpha + (c+a)\cos\beta = a + b + c$.

21. (Stewart's theorem) Let M be any point on side BC. Then $\left(\overline{AM}\right)^2 = \dfrac{b^2 \cdot \overline{MB} + c^2 \cdot \overline{MC}}{a} - \overline{MB} \cdot \overline{MC}$.

22. In any triangle, $m_a^2 = \dfrac{b^2+c^2}{2} - \dfrac{a^2}{4}$.

23. In any triangle, $i_a^2 = bc\left(1 - \left(\dfrac{a}{b+c}\right)^2\right)$.

24. In any triangle, $m_a^2 + m_b^2 + m_c^2 = \dfrac{3}{4}\left(a^2+b^2+c^2\right)$.

25. In any triangle, $m_a^4 + m_b^4 + m_c^4 = \dfrac{9}{16}\left(a^4+b^4+c^4\right)$.

26. In any triangle, if $b^2+c^2 = 2a^2$, then $m_b^2 + m_c^2 = 2m_a^2$ and $cm_c = bm_b$.

27. In any triangle, if $\alpha \neq 90°$, then $8A = \left(4m_a^2 - a^2\right)\tan\alpha$.

28. If $i_a = \dfrac{\sqrt{2}\cdot bc}{b+c}$, then the triangle has a right angle.

29. If $\dfrac{b+c}{a} = \dfrac{h_a}{i_a}\sqrt{2}$, then the triangle has a right angle.

30. If $\dfrac{b^3+c^3-a^3}{b+c-a} = a^2$, then $\alpha = 60°$.

31. If $\dfrac{1}{a+b} + \dfrac{1}{b+c} = \dfrac{3}{a+b+c}$, then $\beta = 60°$.

32. If $c\,(a+b-c) - a\,(b+c-a) + b\,(a+c-b) = bc$, then $\alpha = 60°$.

33. Find the exact value of the area of the triangle in which $a = \sqrt{6}$, $\alpha = 60°$, and $b+c = 3+\sqrt{3}$.

34. If $(b+c)\,a = b^2+c^2$ and $b+c = \dfrac{2\sqrt{3}}{3}a$, then $\alpha = 30°$.

35. If $b^2 - c^2 = 2a^2$, then $\alpha \leq 30°$.

36. If $b^2 + c^2 = 2a^2$, then $\alpha \leq 60°$.

37. If $a < \dfrac{b+c}{2}$, then $\alpha < \dfrac{\beta+\gamma}{2}$.

38. If $b+c = 2a$ and $\beta+\gamma = 2\alpha$, then the triangle is equilateral.

39. In any triangle, $\dfrac{a}{b} = \dfrac{c + \cos\alpha\,(a\cos\gamma - c\cos\alpha)}{c\cos\gamma + a\cos\alpha}$.

40. In any triangle, $\sin^2\alpha + 2\sin\beta\sin\gamma\cos\alpha = \sin^2\beta + \sin^2\gamma$.

41. In any triangle, $4R^2\left(\sin^2\beta + \sin^2\gamma - \sin^2\alpha\right) = 2bc\cos\alpha$.

42. If $\sin\alpha = 2\sin\beta\cos\gamma$, then the triangle is isosceles.

43. If $\sin\alpha = \dfrac{\sin\beta+\sin\gamma}{\cos\beta+\cos\gamma}$, then the triangle has a right angle.

44. In any triangle, $\dfrac{bc\cos\alpha + ac\cos\beta + ab\cos\gamma}{a\sin\alpha + b\sin\beta + c\sin\gamma} = R$.

45. In any triangle, $\dfrac{\sin\alpha+\sin\beta+\sin\gamma}{\cos\alpha+\cos\beta+\cos\gamma}=\dfrac{P}{2(R+r)}$.

46. If $E_1=\dfrac{a^2\sin\beta\cos\gamma}{\sin\alpha}+\dfrac{b^2\cos\alpha\sin\gamma}{\sin\beta}+\dfrac{c^2\sin\alpha\cos\beta}{\sin\gamma}$ and $E_2=\dfrac{a^2\cos\beta\sin\gamma}{\sin\alpha}+\dfrac{b^2\sin\alpha\cos\gamma}{\sin\beta}+\dfrac{c^2\cos\alpha\sin\beta}{\sin\gamma}$, then $E_1=E_2$.

47. In any triangle, $\cot\alpha+\cot\beta+\cot\gamma=\dfrac{R\left(a^2+b^2+c^2\right)}{abc}$.

48. In any triangle, $a\cot\alpha+b\cot\beta+c\cot\gamma=2(R+r)$.

49. In any triangle, $a\cos\alpha+b\cos\beta+c\cos\gamma=\dfrac{2A}{R}$.

50. In any triangle, $\cos\alpha+\cos\beta+\cos\gamma=1+\dfrac{r}{R}$.

51. In any triangle, $16A^2=2\left(a^2b^2+a^2c^2+b^2c^2\right)-\left(a^4+b^4+c^4\right)$.

52. If $b^2+c^2=2a^2$, then $8A=\sqrt{16b^2c^2-\left(b^2+c^2\right)^2}$.

53. If $\alpha\neq 90°$, $\beta\neq 90°$, and $a^2\tan\beta=b^2\tan\alpha$, then the triangle is isosceles or has a right angle.

54. If $\alpha\neq 90°$, then $\left(b^2+c^2-a^2\right)\tan\alpha=4A$.

55. If $\alpha\neq 90°$, $\beta\neq 90°$, then $\left(b^2+c^2-a^2\right)\tan\alpha=\left(a^2+c^2-b^2\right)\tan\beta$.

56. In any triangle, $a(b\cos\beta+c\cos\gamma)=\sin\alpha\left(b^2\cot\beta+c^2\cot\gamma\right)$.

57. In any triangle, $h_a=2R\sin\beta\sin\gamma$.

58. In any triangle, $a^2\cot\alpha+b^2\cot\beta+c^2\cot\gamma=4A$.

59. If $2h_a=a\tan\beta$ $(\beta\neq 90°)$, then the triangle is isosceles.

60. If $\cot\alpha+\cot\beta=2\cot\gamma$, then $a^2+b^2=2c^2$.

61. If $\cot\alpha=\cot\beta+\cot\gamma$, then $b^2+c^2=3a^2$.

62. If $\cos^2\alpha+\cos^2\beta+\cos^2\gamma=1$, then the triangle has a right angle.

63. If $\sin^2\alpha+\sin^2\beta+\sin^2\gamma=2$, then the triangle has a right angle.

64. If $a^2+b^2+c^2=8R^2$, then the triangle has a right angle.

65. If $b+c=2a$, then $2\cos\alpha+\cos\beta+\cos\gamma=2$.

66. If $5a^2=5b^2+c^2$ and $3b^2=3c^2+a^2$, then $\tan\alpha=3$, $\tan\beta=2$, and $\tan\gamma=1$.

67. If $\dfrac{\sin\gamma}{\sin\beta}+\dfrac{\sin\beta}{\sin\gamma}=2\sin\alpha$, then the triangle is isosceles with a right angle.

68. If $A=\dfrac{a^2+b^2}{4}$, then the triangle is isosceles with a right angle.

69. What can we state about a triangle in which $\cos\alpha+\cos\beta+\cos\gamma=1$?

70. If $\dfrac{P}{2}=b\cos\alpha+c\cos\beta+a\cos\gamma$, then the triangle is equilateral.