Until now, we have studied objects' motion along a straight line. Many objects, however, are move along a circular path. Right now, as we speak, if we are on Earth, then we are traveling around the Sun along a path that can be approximated as a circle.

If we think about it, lots of things move along a circular path. Planets and moons orbit around the Sun in our solar system. Particles motion can often be modeled as orbiting another particle. Or just rotate a bicycle wheel and watch a point on it moving around. Indeed, many things move along circular paths and not along a straight line. This new situation will bring new challenges and new techniques.

Example 1. Two runners are training together. They use a circular lap in the stadium. Andy runs at a speed of $9.6\ \frac{\mathrm{mi}}{\mathrm{h}}$ and Brandon runs at a speed of $10.5\ \frac{\mathrm{mi}}{\mathrm{h}}$. Andy and Brandon start at the same time, at the same point, running in the same direction. How long is the circular path if Brandon catches up with Andy after 18 minutes of running?

Solution: When they start running, Brandon will get ahead as his speed is greater. If they were to run along a straight line, the distance between them would just increase with time. That's not what happens with a circular path. Eventually Brandon will catch up with Andy when the difference between the distances traveled is the length of a complete lap.

Let us first convert 18 minutes to hours. We will use a conversion factor.

$$18\ \mathrm{min} = \frac{18\ \cancel{\mathrm{min}}}{1} \cdot \frac{1\ \mathrm{h}}{60\ \cancel{\mathrm{min}}} = \frac{18}{60}\ \mathrm{h} = 0.3\ \mathrm{h}$$

Now we can compute the distance traveled by Andy and Brandon:

$$s_A = v_A \cdot t = 9.6\ \frac{\mathrm{mi}}{\cancel{\mathrm{h}}} \cdot 0.3\ \cancel{\mathrm{h}} = 2.88\ \mathrm{mi} \qquad \text{and} \qquad s_B = v_B \cdot t = 10.5\ \frac{\mathrm{mi}}{\cancel{\mathrm{h}}} \cdot 0.3\ \cancel{\mathrm{h}} = 3.15\ \mathrm{mi}$$

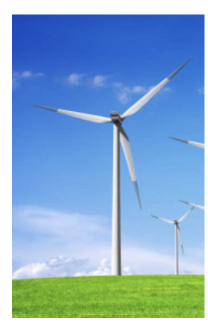
The length of a lap is the difference between the distances traveled:

$$L = s_B - s_A = 3.15\ \mathrm{mi} - 2.88\ \mathrm{mi} = \boxed{0.27\ \mathrm{mi}}$$

A similar example is the race between the minute and hour hands on a watch. At what time after 12:00 noon do the two hands meet again for the first time?

What is interesting here is that this question cannot be solved using the techniques we just saw because we have no information in terms of speed and distance traveled. But we suspect that the result should not depend on the size of the watch or the size of its hands. Therefore, we need some new concepts to address this question.

Consider a blade of a wind turbine. As the blade rotates, each point on it is moving along a circle. The circles are concentric (i.e have the same center). A point near the center is rotating on a much smaller circle, while a point near the top is rotating on a larger circle. The traditional concept of speed, the distance traveled per time can be applied to each point individually, but the farther the point to the center is, the greater the speed is. Therefore, the traditional concept of speed cannot describe the entire blade.



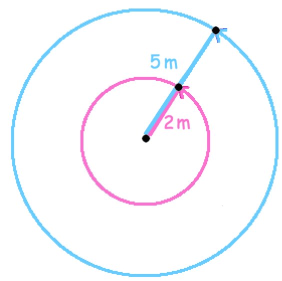
Example 2. The blade of a wind turbine completes a rotation in 30 seconds. Find the speed, measured in meters per second, of a point on the blade

a) 2 meters from the center b) 5 meters from the center

Solution: Speed is defined as distance traveled divided by the time. The time is given as 30 seconds, and the distance is the circumference of a circle. In the first case, the radius of the circle is 2 meters, in the second case it is 5 meters.

$$v_1 = \frac{\text{distance}}{\text{time}} = \frac{2\pi r_1}{t} = \frac{2\pi\,(2\text{ m})}{30\text{ s}} \approx 0.41888\,\frac{\text{m}}{\text{s}}$$

$$v_2 = \frac{\text{distance}}{\text{time}} = \frac{2\pi r_2}{t} = \frac{2\pi\,(5\text{ m})}{30\text{ s}} \approx 1.0472\,\frac{\text{m}}{\text{s}}$$



The traditional concept of speed works for parts of the turbine blade, but not for the entire thing. Yet, the blade is not falling apart, there is something common about the motion of all parts, and that is what we call **angular speed**.

Let us first review the regular, linear speed: the speed is denoted by v and it measures the distance traveled per time. If time is denoted by t, distance by s, then

$$v = \frac{s}{t}, \qquad s = v \cdot t, \quad \text{and} \quad t = \frac{s}{v}$$

Speed can be measured in various units, but it is always a quotient of distance and time. $50\,\frac{\text{mi}}{\text{hr}}$ means that the object travels 50 miles in 1 hour. $3000\frac{\text{km}}{\text{s}}$ means that the object travels 3000 km in a second, and so on.

Angular velocity, denoted by ω works similarly, only distance traveled is replaced by the angle traveled through, denoted by θ.

$$\omega = \frac{\theta}{t}, \qquad \theta = \omega \cdot t, \quad \text{and} \quad t = \frac{\theta}{\omega}$$

Angular speed can be measured in various units, and it is usually a quotient of an angle and time. $28\frac{\text{deg}}{\text{min}}$ would mean that the object rotates through $28°$ in a minute. Another way to express angular speed is to express the number of rotations per time.

Example 3. a) Suppose that the Earth revolves around the Sun along a circular path. What is its angular speed?

b) A rotating light completes 8 rotations per minute. What is its angular speed? How long does it take for the light to rotate through an angle of $50°$?

Solution: a) Angular velocity is the angle traveled divided by the time. We know that the Earth completes a full cycle in each year. Therefore, in simplest terms, the angular velocity is $\dfrac{360°}{1\text{ year}}$

$$\omega = \frac{\theta}{t} = \frac{360°}{1\text{yr}} = \frac{360°}{1\text{ year}}$$

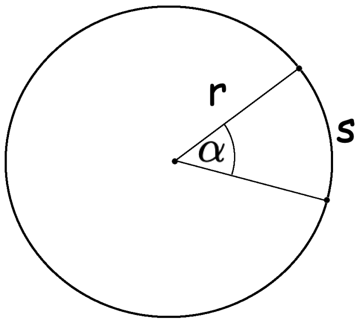
b) Rotations can be easily converted to angles. Numbers might be more pleasant if we measure time in seconds, instead of in minutes.

$$\omega = \frac{\theta}{t} = \frac{8 \cdot 360^\circ}{60\text{s}} = \frac{480^\circ}{1\text{ s}} = 480\frac{\text{deg}^\circ}{\text{s}}$$

$$t = \frac{\theta}{\omega} = \frac{50^\circ}{\frac{480^\circ}{1\text{ s}}} \approx 0.10416\text{ s}$$

However, in case of angular speed, we have a very good reason to prefer the use radian measure of angles. Recall the definition of radian measure of angles.

Definition: Suppose that α is an angle between 0° and 360°. If we draw this angle from the center of any circle, the radian measure of the angle is defined as the ratio of the arc length determined by α and the radius in the circle.



Given this definition, we can convert 360° to radians because the arc belonging to this angle is the entire circumference of the circle. For any radius r,

$$360^\circ = \frac{s}{r} = \frac{2\pi r}{r} = 2\pi\text{ rad}$$

If we divide both sides by 2, we get that $180^\circ = \pi$ rad. Using this equation and proportions, we can now convert any angle to radians, not just the ones between 0° and 360°. For example, $90^\circ = \frac{\pi}{2}$ rad, $45^\circ = \frac{\pi}{4}$ rad, $60^\circ = \frac{\pi}{3}$ rad, and so on.

To convert between radians and degrees, we can use the conversion factors $\frac{180^\circ}{\pi\text{ rad}}$ and $\frac{\pi\text{ rad}}{180^\circ}$.

Example 4. Perform the given conversions.

a) 1020° to radians. b) $\frac{5\pi}{4}$ radians to degrees. c) 1 radian to degrees.

Solution: a) We will use conversion factors. These are fractions of value 1. We select the fraction that will cancel out what we don't want and brings in what we do want.

$$1020^\circ = \frac{1020^\circ}{1} \cdot \frac{\pi\text{ rad}}{180^\circ} = \frac{1020}{180}\pi\text{ rad} = 6\pi\text{ rad}$$

b) We will now use the conversion factor $\dfrac{180^\circ}{\pi\ \mathrm{rad}}$

$$\frac{5\pi}{4}\ \mathrm{rad} = \frac{5\,\cancel{\pi\ \mathrm{rad}}}{4}\cdot\frac{180^\circ}{\cancel{\pi\ \mathrm{rad}}} = \frac{5}{4}\cdot 180^\circ = 225^\circ$$

We don't always need to use the conversion factor. We can just simply replace π by 180° in the expression $\dfrac{5\pi\ \mathrm{rad}}{4}$ and simplify the fraction.

c) We will use the same conversion factor as before.

$$1\ \mathrm{rad} = 1\,\cancel{\mathrm{rad}}\cdot\frac{180^\circ}{\pi\,\cancel{\mathrm{rad}}} = \left(\frac{180}{\pi}\right)^\circ \approx 57.29578^\circ$$

Discussion: Draw a circle and a central angle of 1 radian. Can you explain what it would mean for an angle to measure 1 radian? Why is it close to 60°? And why it has to be slightly less than 60°?

Let us also note that radian is a unit-less unit, just a real number, as we divide a length by another length. For this reason, $5\ \dfrac{\mathrm{rad}}{\mathrm{s}}$ will also often be denoted by $5\ \dfrac{1}{\mathrm{s}}$.

Converting between different units of angular speed is just a matter of converting both numerator and denominator.

Example 5. Perform each of the given conversions.

a) Convert 32 rpm (rotation per minute) to radians per second.

b) Convert $150\dfrac{1}{s}$ to rpm, rotations per minute.

Solution: a) We will separately convert the numerator and denominator. 1 rotation $= 2\pi$ radians and 1 minute is 60 seconds.

$$32\ \mathrm{rpm} = 32\ \frac{\mathrm{rotations}}{1\ \mathrm{min}} = 32\frac{2\pi}{60\ \mathrm{s}} = \frac{64\pi}{60}\ \frac{1}{\mathrm{s}} = \frac{16}{15}\pi\ \frac{1}{\mathrm{s}} \approx 3.351\ \frac{1}{\mathrm{s}}$$

Note that radian is a unit-less unit, and so $\dfrac{\mathrm{rad}}{\mathrm{s}}$ is the same as $\dfrac{1}{\mathrm{s}}$. It is just a courtesy of the writer to explicitly indicate the radian in any measurement.

b) We will separately convert the numerator and denominator. 1 rotation $= 2\pi$ radians and 1 minute is 60 seconds. This time we will use conversion factors, one for the numerator, one for the denominator.

$$150\ \frac{1}{\mathrm{s}} = 150\ \frac{1\cancel{\mathrm{rad}}}{\cancel{\mathrm{s}}}\cdot\frac{1\ \mathrm{rotation}}{2\pi\cancel{\mathrm{rad}}}\cdot\frac{60\ \cancel{\mathrm{s}}}{1\ \mathrm{min}} = \frac{150\cdot 60}{2\pi}\ \frac{\mathrm{rot}}{\mathrm{min}} \approx 1432.3945\ \frac{\mathrm{rot}}{\mathrm{min}}$$

Here is why it is beneficial to use radians in problems involving angular speed. Because of its definition, if angles are expressed in radians, then the distance traveled is the angle traveled through times the radius, and the linear speed is just the angular speed times the radius. In short, $s = \theta r$ and $v = \omega r$.

Indeed, if $\theta_{\text{rad}} = \dfrac{s}{r}$, then $s = \theta_{\text{rad}} \cdot r$ and $v = \dfrac{s}{t} = \dfrac{\theta_{\text{rad}} r}{t} = r \cdot \dfrac{\theta_{\text{rad}}}{t} = r \cdot \omega$

Example 6. The blade of a wind turbine rotates with an angular speed of $1.2 \frac{1}{\mathrm{s}}$.

a) Find the speed, measured in meters per second, of a point on the blade 2 meters from the center and 5 meters from the center.

b) Find the angle traveled through in 18 minutes. Find the distance traveled by a point on the blade 2 meters from the center and 5 meters from the center.

Solution: a) We will apply our nice simple formula $v = \omega r$.

$$v_1 = \omega \cdot r_1 = 1.2 \, \frac{1}{\mathrm{s}} \cdot 2 \mathrm{~m} = 2.4 \, \frac{\mathrm{m}}{\mathrm{s}} \quad \text{and} \quad v_2 = \omega \cdot r_2 = 1.2 \, \frac{1}{\mathrm{s}} \cdot 5 \mathrm{~m} = 6 \, \frac{\mathrm{m}}{\mathrm{s}}$$

b) We will express 18 minutes in seconds. $18 \mathrm{~min} = 18 \cdot 60 \mathrm{~s} = 1080 \mathrm{~s}$.

$$\theta = \omega t = 1.2 \, \frac{1}{\cancel{\mathrm{s}}} \cdot 1080 \cancel{\mathrm{s}} = 1296 \ \ (\mathrm{rad})$$

For the distance traveled, we will apply our nice simple formula $s = r\theta$.

$$s_1 = r_1 \cdot \theta = 2 \mathrm{~m} \cdot 1296 = 2592 \mathrm{~m} \quad \text{and} \quad s_2 = r_2 \cdot \theta = 5 \mathrm{~m} \cdot 1296 = 6480 \mathrm{~m}$$

Practice Problems

1. Two runners, Ade and Jake are training together on a circular track 420 meters long. Ade runs at a speed of $3.5 \, \frac{\mathrm{m}}{\mathrm{s}}$ and Jake at $4.7 \, \frac{\mathrm{m}}{\mathrm{s}}$. They start running at the same, at the same point.

   a) How long until they meet again if they run in the same direction?

   b) How long until they meet again if they run in opposite directions?

2. An old vinyl record is played on a turntable rotating at a rate of 45 rotations per minute.

   a) Find the angular speed in radians per second.

   b) Find the linear speed of a point on the record 0.12 meters from the center of rotation.

3. Let us assume that the earth is shaped as a sphere of radius 3960 miles. The earth rotates about its axis and completes a rotation every day. Assuming no other motion, what is the greatest possible linear velocity on any point of the surface of Earth?

Answers

1. a) 5 minutes and 50 seconds b) 51.22 seconds 2. a) $4.7124\ \frac{1}{\mathrm{s}}$ b) $0.5655\ \frac{\mathrm{m}}{\mathrm{s}}$ 3. $1036.7256\ \frac{\mathrm{mi}}{\mathrm{h}}$

