Theorem: If a circle has radius r, then its circumference and area can be computed as

$$C = 2\pi r \quad \text{and} \quad A = \pi r^2$$

Sample Problems

1. The radius of a circle is $8\mathrm{m}$.

   a) Find the length of an arc subtended by a central angle of $20°$.

   b) Find the area of a sector subtended by a central angle of $20°$.

2. Phoenix, AZ and Salt Lake City, UT have approximately the same longitude. The radius of the earth is approximately 3960 miles. The latitude of Phoenix is $33.5°$ and that of Salt Lake City is $40.7°$. Find the distance to the nearest mile between the two cities.

3. Find the radius of a circle if we know that a sector subtended by a central angle of $67°$ has an area of $42\mathrm{m}^2$.

4. The minute hand of a clock is 5 cm long. Find the speed of the top of the minute hand. Express your answer in meter per second.

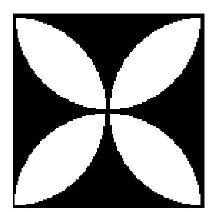
5. A satellite can be seen over the same point on Earth above the equator. It is 200 miles above the surface. Find the speed of the satellite in miles per hour. (The radius of the earth is 3960 mi).

6. Consider the cricles C_1 and C_2. An arc subtended by a central angle of $40°$ in C_1 is has the same length as an arc in C_2 that is subtended by a central angle of $60°$.

   a) Find the ratio between the radii of C_1 to C_2.

   b) Find the ratio between the areas of C_1 to C_2.

7. Consider the image shown. There are four semicircles in a square, each with a radius of 1 unit. Find the exact and approximate value of the shaded area.



Practice Problems

1. The radius of a circle is $20\mathrm{m}$.

   a) Find the length of an arc subtended by a central angle of $58°$.

   b) Find the area of a sector subtended by a central angle of $58°$.

2. Seattle, WA and San Fracisco, CA have approximately the same longitude. The radius of the earth is approximately 3960 miles. The latitude of Seattle is $47.67°$ and that of San Fracisco is $37.83°$. Find the distance to the nearest mile between the two cities.

3. Find the radius of a circle if we know that a sector subtended by a central angle of $32°$ has an area of $5\mathrm{m}^2$.

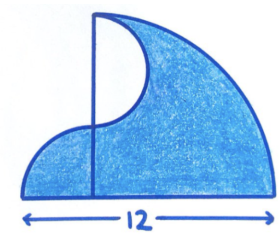
4. The hour hand of a clock is $4\mathrm{cm}$ long. Find the speed of the top of the hour hand. Express your answer in meter per second.

5. A satellite can be seen over the same point on Earth above the equator. It is 300 miles above the surface. Find the speed of the satellite in miles per hour. (The radius of the earth is 3960 mi).

6. Consider the circles C_1 and C_2. An arc subtended by a central angle of $80°$ in C_1 is has the same length as an arc in C_2 that is subtended by a central angle of $48°$.

   a) Find the ratio between the radii of C_1 to C_2.
   b) Find the ratio between the areas of C_1 to C_2.

7. This problem is from the great Catriona Shearer. The picture shows two quarter circles and a semicircle. What is the shaded area?



Enrichment

1. Suppose the earth were a perfect sphere with a perfectly fitting belt of $24\,000$ miles surrounding it along a great circular path. Suppose the belt was cut, and one hundred feet of additional material was added to the belt, with the "loose fit" evenly distributed around the earth so that the new belt was still circular with its center at the center of the earth. Which of the following best describes the resulting situation?

   A) We could slip a piece of paper between the belt and the earth. B) We could get our fingers under the belt.
   C) We could crawl under the belt.
   D) We could walk upright under the belt.
   E) We could drive a truck under the belt.

Answers

Sample Problems

1. a) $2.793\mathrm{m}$ b) $11.170\mathrm{m}^2$ 2. 498 miles 3. $8.475\,5\mathrm{m}$

4. $0.0000873\dfrac{\mathrm{m}}{\mathrm{s}} = 8.73\times 10^{-5}\dfrac{\mathrm{m}}{\mathrm{s}}$ 5. $1089.085\,45\dfrac{\mathrm{mi}}{\mathrm{h}}$ 6. a) $\dfrac{r_1}{r_2} = \dfrac{3}{2}$ b) $\dfrac{A_1}{A_2} = \dfrac{9}{4}$ 7. $8 - 2\pi \approx 1.716\,8$ (unit^2)

Practice Problems

1. a) $20.245\,82\mathrm{m}$ b) $202.458\,2\mathrm{m}^2$ 2.) 680 miles 3.) $4.231\,42\mathrm{m}$

4. $0.000005817\,76\dfrac{\mathrm{m}}{\mathrm{s}} = 5.817\,76\times 10^{-6}\dfrac{\mathrm{m}}{\mathrm{s}}$ 5. $1115.265\,4\dfrac{\mathrm{mi}}{\mathrm{h}}$ 6. a) $\dfrac{r_1}{r_2} = \dfrac{3}{5}$ b) $\dfrac{A_1}{A_2} = \dfrac{9}{25}$ 7. 18π

Sample Problems Solutions

1. The radius of a circle is $8\mathrm{m}$.

   a) Find the length of an arc subtended by a central angle of $20°$.

   Solution: The circumference of the circle is $C = 2\pi r = 2\pi(8\mathrm{m}) = 16\pi\mathrm{m}$. This is the arc belonging to the central angle of $360°$. If we wanted to find the arc length belonging to a central angle of $1°$, we would just need to divide the circumference by 360. To obtain the arc length belonging to a central angle of $20°$, we would need to take the $1°$degree arc twenty times.

$$x = \frac{C}{360}(20) = \frac{C}{18} = \frac{16\pi\ \mathrm{m}}{18} = \frac{8}{9}\pi\ \mathrm{m} \approx 2.793\ \mathrm{m}$$

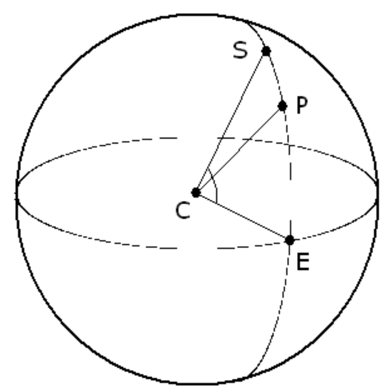
   b) Find the area of a sector subtended by a central angle of $20°$.

   Solution: The area of the circle is $A = \pi r^2 = \pi(8\mathrm{m})^2 = 64\pi\mathrm{m}^2$. This is the area of the sector belonging to the central angle of $360°$. If we wanted to find the area of a sector with a central angle of $1°$, we would just need to divide the area of the circle by 360. To obtain the area of a sector with a central angle of $20°$, we would need to take the $1°$degree sector twenty times.

$$x = \frac{A}{360}(20) = \frac{A}{18} = \frac{64\pi\mathrm{m}^2}{18} = \frac{32}{9}\pi\mathrm{m}^2 \cong 11.170\ \mathrm{m}^2$$

2. Phoenix, AZ and Salt Lake City, UT have approximately the same longitude. The radius of the earth is approximately 3960 miles. The latitude of Phoenix is $33.5°$ and that of Salt Lake City is $40.7°$. Find the distance to the nearest mile between the two cities.

   Solution: This is just an arc length problem. Consider the picture shown. Let E be the point on the equator located on the same longitude as Salt Lake City (point S) and Phoenix (point P).
   It is given that angle $SCE = 40.7°$ and angle $PCE = 33.5°$. The distance between the cities is the arc length SP, belonging to the central angle $40.7° - 33.5° = 7.2°$.



$$s = \frac{C}{360}(7.2) = \frac{2\pi R(7.2)}{360} = \frac{2\pi(3960\mathrm{mi})(7.2)}{360} \approx 497.628\,276\mathrm{mi}$$

   We round this result to 498 miles.

3. Find the radius of a circle if we know that a sector subtended by a central angle of $67°$ has an area of $42\mathrm{m}^2$.

   Solution: The ratio between the area of the circle and the sector is the same as the ratio of $360°$ to $67°$. Also, recall that the area of a circle with radius r is $A = \pi r^2$.

$$\begin{aligned}\frac{\pi r^2}{42\mathrm{m}^2} &= \frac{360°}{67°} \qquad \text{solve for } r\\ r^2 &= \frac{360°\left(42\mathrm{m}^2\right)}{67°(\pi)}\\ r &= \sqrt{\frac{360\cdot 42\mathrm{m}^2}{67\pi}} = \sqrt{\frac{15\,120}{67\pi}}\mathrm{m} \approx 8.475\,5\mathrm{m}\end{aligned}$$

4. The minute hand of a clock is 5 cm long. Find the speed of the top of the minute hand. Express your answer in meter per second.

Solution: The minute hand travels a full circle in one hour. (Note that $1\mathrm{cm} = 0.01\mathrm{m}$)

$$\text{speed} = \frac{\text{distance}}{\text{time}} = \frac{2\pi r}{t} = \frac{2\pi\,(5\mathrm{cm})}{1\mathrm{hr}} = \frac{10\pi\mathrm{cm}}{3600\mathrm{s}} = \frac{10\pi\,(0.01\mathrm{m})}{3600\mathrm{s}} = \frac{0.1\pi}{3600}\frac{\mathrm{m}}{\mathrm{s}} = 8.727\times 10^{-5}\frac{\mathrm{m}}{\mathrm{s}}$$

5. A satellite can be seen over the same point on Earth above the equator. It is 200 miles above the surface. Find the speed of the satellite in miles per hour. (The radius of the earth is 3960 mi).

Solution: If the satellite appears to be at the same point, it must travel along with the Earth, covering a full circle in exactly one day. The radius of the circle is the sum of the radius of the Earth and the distance from the surface.

$$\text{speed} = \frac{\text{distance}}{\text{time}} = \frac{2\pi\,(3960\mathrm{mi} + 200\mathrm{mi})}{24\mathrm{h}} = \frac{1040\pi}{3}\frac{\mathrm{mi}}{\mathrm{h}} \approx 1089.08545\frac{\mathrm{mi}}{\mathrm{h}}$$

6. Consider the cricles C_1 and C_2. An arc subtended by a central angle of $40°$ in C_1 is has the same length as an arc in C_2 that is subtended by a central angle of $60°$.

a) Find the ratio between the radii of C_1 to C_2. b) Find the ratio between the areas of C_1 to C_2.

Solution: Let r_1 and r_2 denote the radii of C_1 and C_2. We write an equation expressing the arc lengths and simplify the equation.

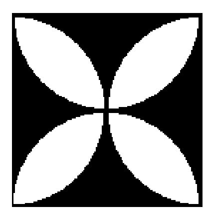
$$\begin{aligned} \frac{2\pi r_1}{360}(40) &= \frac{2\pi r_2}{360}(60) && \text{divide by } 2\pi\text{, multiply by } 360 \\ 40r_1 &= 60r_2 && \text{divide by } 20 \\ 2r_1 &= 3r_2 && \text{divide by } 2r_2 \\ \frac{r_1}{r_2} &= \frac{3}{2} && \text{Thus the ratio of } r_1 \text{ to } r_2 \text{ is 3 to 2.} \end{aligned}$$

b) We express the ratio between the ares and hope that we can relate that to the ratio of the radii we found in part a)

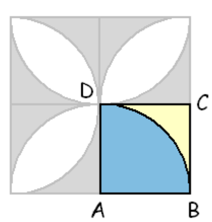
$$\frac{A_1}{A_2} = \frac{\pi r_1^2}{\pi r_2^2} = \frac{r_1^2}{r_2^2} = \left(\frac{r_1}{r_2}\right)^2 = \left(\frac{3}{2}\right)^2 = \frac{9}{4}$$

So the ratio between the areas is $\boxed{9 \text{ to } 4}$.

7. Consider the image shown. There are four semicircles in a square, each with a radius of 1 unit. Find the exact and approximate value of the shaded area. Solution: Square ABCD has an area of 1 unit^2. Sector ABD is subtended by a central angle of $90°$. Therefore, its area is exactly $\frac{1}{4}$ of the area of a full circle, that is,



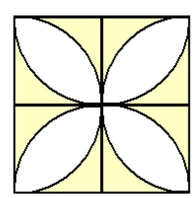
$$A_{\text{sector}} = \frac{1}{4}\pi\,(1)^2 = \frac{\pi}{4}\,(\text{unit}^2)$$



The area shaded with yellow is the difference of the two:

$$A_{\text{yellow}} = 1 - \frac{\pi}{4}\ (\text{unit}^2)$$

The next image shows that the area we need to find is composed of the same shape, repeated eight times. Therefore, the total shaded area we are looking for is



$$A = 8\left(1 - \frac{\pi}{4}\right) = \boxed{8 - 2\pi \approx 1.7168\,(\text{unit}^2)}$$

