Sample Problems

1. I am thinking of an angle α. If twice α is co-terminal to 100°, does that mean that α is co-terminal to 50°?

2. Three times an angle β is co-terminal to 120°. Then β is co-terminal to what angle?

3. Consider the equation $\sin 2x = \dfrac{1}{2}$.

   a) Solve the equation and present all solutions in degrees.
   b) Find all solutions of the equation that fall between 0° and 360°.
   c) Draw a picture of the solutions between 0° and 360°.

4. Consider the equation $\cos 3x = -\dfrac{\sqrt{3}}{2}$.

   a) Find all solutions for the equation.
   b) Find all solutions for the equation that fall between 0° and 360°. Present these angles in degrees.
   c) Draw a picture of the solutions between 0° and 360°.

5. Consider the equation $\tan 5x = -1$.

   a) Find all solutions for the equation. Present your answer in degrees.
   b) Find all solutions for the equation. Present your answer in radians.
   c) Find all solutions for the equation that fall between 0° and 360°.
   d) Draw a picture of the solutions between 0° and 360°.

6. a) Solve the equation $-\sin 5x = \cos 10x$
   b) List all solutions (in degrees) that fall between 0° and 360°.

Practice Problems

1. Consider the equation $\sin 3x = -\dfrac{\sqrt{3}}{2}$.

   a) Solve the equation. Present all solutions in degrees.
   b) Solve the equation. Present all solutions in radians.
   c) List all solutions between 0° and 360°.
   d) Draw a picture of the solutions between 0° and 360°.

2. Find the exact value of all solutions for each of the following equations. Present your answer in radians.

   a) $\tan 6x = -\dfrac{1}{\sqrt{3}}$ b) $\sin 3x = \cos 3x$ c) $\sin 4x = -1$

3. List all solutions of the equation $\sin 3x = -\dfrac{1}{2}$ that are between 0° and 360°.

Sample Problems - Answers

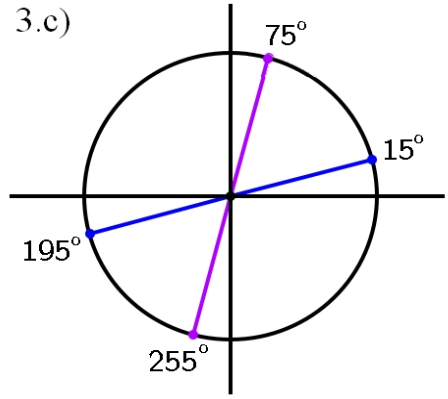
1. Not necessarily. α is co-terminal to either 50° or to 230°.

2. β is co-terminal to either 40°, or 160°, or 280°

3. a) $x = 15^\circ + k \cdot 180^\circ$
   or $x = 75^\circ + k \cdot 180^\circ$
   where $k \in \mathbb{Z}$

   b) 15°, 75°, 195°, and 255°

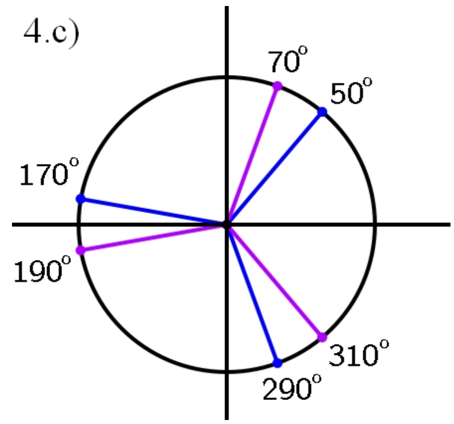
3.c)



4. a) in degrees: $x = \pm 50^\circ + k \cdot 120^\circ$, where $k \in \mathbb{Z}$
   in radians: $x = \pm \dfrac{5\pi}{18} + k\dfrac{2\pi}{3}, \quad k \in \mathbb{Z}$

   b) $50^\circ, 70^\circ, 170^\circ, 190^\circ, 290^\circ, 310^\circ$

4.c)

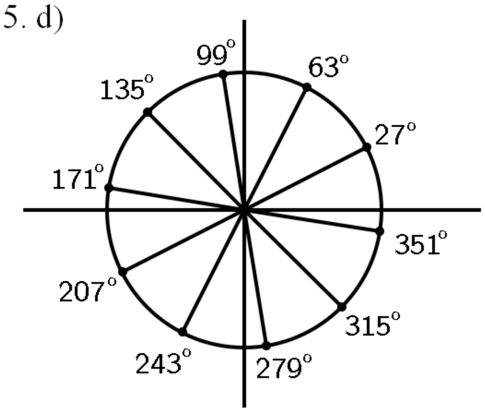


5. a) $x = -9^\circ + k \cdot 36^\circ$ where $k \in \mathbb{Z}$

   b) $x = -\dfrac{\pi}{20} + k\dfrac{\pi}{5} \qquad k \in \mathbb{Z}$

   c) 27°, 63°, 99°, 135°, 171°, 207°, 243°, 279°, 315°, and 351°

5. d)



6. a) in degrees: $x = -6^\circ + k \cdot 72^\circ$ or $x = -25^\circ + k \cdot 72^\circ$ or $x = -18^\circ + k \cdot 72^\circ$ where $k \in \mathbb{Z}$

   in radians: $x = -\dfrac{\pi}{30} + k \cdot \dfrac{2\pi}{5}$ or $x = -\dfrac{5\pi}{36} + k \cdot \dfrac{2\pi}{5}$ or $x = -\dfrac{\pi}{10} + k \cdot \dfrac{2\pi}{5}$ where $k \in \mathbb{Z}$

   b) $47^\circ, 54^\circ, 66^\circ, 119^\circ, 126^\circ, 138^\circ, 191^\circ, 198^\circ, 210^\circ, 263^\circ, 270^\circ, 282^\circ, 335^\circ, 342^\circ, 354^\circ$

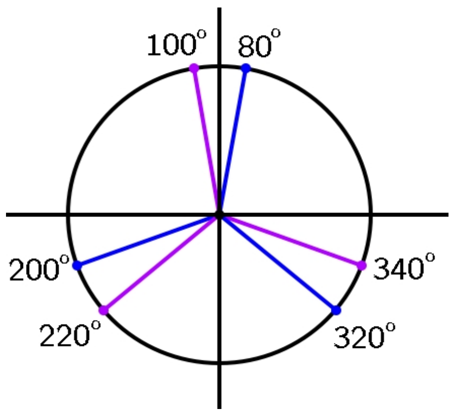
Practice Problems - Answers

1. a) $x = -20° + k \cdot 120°$ or $x = -40° + k \cdot 120°$ where $k \in \mathbb{Z}$

   b) $x = -\dfrac{\pi}{9} + k \cdot \dfrac{2\pi}{3}$ or $x = -\dfrac{2\pi}{9} + k \cdot \dfrac{2\pi}{3}$ where $k \in \mathbb{Z}$

   c) 80°, 100°, 200°, 220°, 320°, 340°

   d)



2. a) $-\dfrac{\pi}{36} + k\dfrac{\pi}{6}$ where $k \in \mathbb{Z}$ b) $\dfrac{\pi}{12} + k\dfrac{\pi}{3}$, $k \in \mathbb{Z}$ c) $-\dfrac{\pi}{8} + k\dfrac{\pi}{2}$, $k \in \mathbb{Z}$

3. $70°, 110°, 190°, 230°, 310°, 350°$

Sample Problems - Solutions

1. I am thinking of an angle α. If twice α is co-terminal to $100°$, does that mean that α is co-terminal to $50°$?

   Solution: Not necessarily. Let us express that twice α is co-terminal to $100°$. Two angles are co-terminal when they differ by a multiple of $360°$:

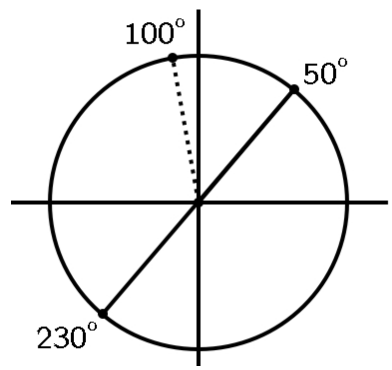
$$\begin{aligned} 2\alpha &= 100° + k \cdot 360° \quad \text{where } k \text{ is an integer} \qquad \text{divide by 2} \\ \alpha &= 50° + k \cdot 180° \end{aligned}$$

   Let us investigate what angles we obtained with the expression $50° + k \cdot 180°$.

   If $k = 0$, then $\alpha = 50° + 0 \cdot 180° = 50°$.
   If $k = 1$, then $\alpha = 50° + 1 \cdot 180° = 230°$.
   If $k = 2$, then $\alpha = 50° + 2 \cdot 180° = 410°$ - co-terminal to $50°$.
   If $k = 3$, then $\alpha = 50° + 3 \cdot 180° = 590°$ - co-terminal to $230°$.
   and so on, all such values are co-terminal to either $50°$ or to $230°$.

   Could α be $230°$? If α is $230°$, then twice α is $460°$ which is indeed co-terminal to $100°$ since $460° = 100° + 360°$.

   So the answer is that α is either co-terminal to $50°$ or to $230°$. These two angles are not co-terminal, they differ by $180°$. But if we double them both, the difference between them becomes $360°$ - so their doubles are co-terminal.

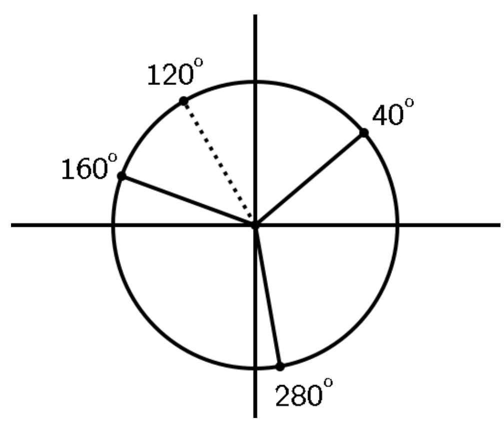


2. Three times an angle β is co-terminal to $120°$. Then β is co-terminal to what angle?

   Solution: We state that three times β is co-terminal to $90°$ and divide both sides by 3.

$$\begin{aligned} 3\beta &= 120° + k \cdot 360° \quad \text{where } k \text{ is an integer} \\ \beta &= 40° + k \cdot 120° \end{aligned}$$

   When $k = 0$, then $\beta = 40°$.
   When $k = 1$, then $\beta = 160°$.
   When $k = 2$, then $\beta = 280°$.
   When $k = 3$, then $\beta = 400°$ - co-terminal to $40°$.
   When $k = 4$, then $\beta = 520°$ - co-terminal to $160°$.



   And so on, all other values will produce angles co-terminal with one of the angles above. So our answer is that β is co-terminal to either $40°$, or $160°$, or $280°$.

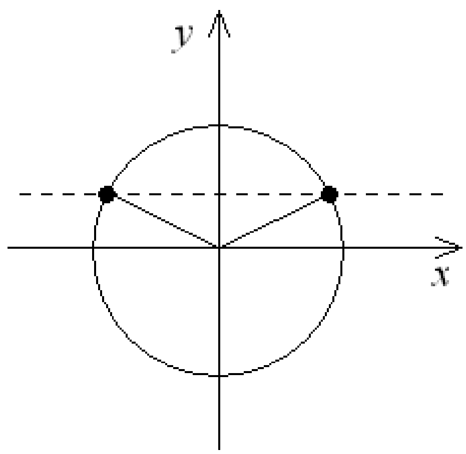
   Does our answer make sense? We could argue that if two angles differ by $120°$, then after multiplying both by 3, the difference becomes $360°$ - so they become co-terminal.

3. Consider the equation $\sin 2x = \dfrac{1}{2}$.

a) Solve the equation and present all solutions in degrees.

Solution: We will first solve for $2x$.

$$\sin 2x = \frac{1}{2}$$



$$2x = 30^\circ + k \cdot 360^\circ \qquad \text{or} \qquad 2x = 150^\circ + k \cdot 360^\circ \quad \text{where } k \in \mathbb{Z}$$

Next we solve for x in both equations by dividing both sides by 2.

$$x = 15^\circ + k \cdot 180^\circ \qquad \text{or} \qquad x = 75^\circ + k \cdot 180^\circ \quad \text{where } k \in \mathbb{Z}$$

b) Find all solutions of the equation that fall between 0° and 360°.

In this case, the picture above contains all these angles. To obtain these angles, we need to consider all suitable integer values of k in the expressions $x = 15^\circ + k \cdot 180^\circ$ and $x = 75^\circ + k \cdot 180^\circ$. First consider $x = 15^\circ + k \cdot 180^\circ$.

k	-1	0	1	2	3	4	5
$x = 15^\circ + k \cdot 180^\circ$	-165°	15°	195°	375°	555°	735°	915°

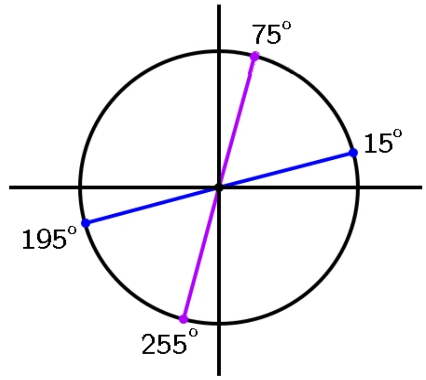
From all these values, 15° and 195° fall between 0° and 360°. Similarly, we consider $x = 75^\circ + k \cdot 180^\circ$

k	-1	0	1	2	3	4	5
$x = 15^\circ + k \cdot 180^\circ$	-105°	75°	255°	435°	615°	795°	975°

From all these values, 75° and 255° fall between 0° and 360°. So the final answer is 15°, 75°, 195°, and 255°.

c) Draw a picture of the solutions between 0° and 360°.

Solution: The first group, $x = 15^\circ + k \cdot 180^\circ$ (k integer) produces angles that are co-terminal with 15° or 195°. The second group, $x = 75^\circ + k \cdot 180^\circ$ (k integer) produces angles that are co-terminal with 75° or 255°.



4. Consider the equation $\cos 3x = -\dfrac{\sqrt{3}}{2}$.

a) Find all solutions for the equation.

Solution: We first solve for $3x$.

$$\begin{aligned} \cos 3x &= -\frac{\sqrt{3}}{2} \\ 3x &= \pm 150^\circ + k\cdot 360^\circ \text{ where } k\in\mathbb{Z} \qquad \text{divide by 3} \end{aligned}$$

We now solve for x by dividing both sides by 3.

$$x = \pm 50^\circ + k\cdot 120^\circ \text{ where } k \text{ is an integer}$$

Finally, we convert the answer to radians

$$\begin{aligned} x &= \pm 50^\circ\left(\frac{\pi}{180^\circ}\right) + k\cdot 120^\circ\left(\frac{\pi}{180^\circ}\right) \text{ where } k \text{ is an integer} \\ x &= \pm\frac{5\pi}{18} + k\cdot\frac{2\pi}{3} \text{ where } k \text{ is an integer.} \end{aligned}$$

b) Find all solutions for the equation that fall between 0° and 360°. Present these angles in degrees.

Solution: Consider the general solution, $x = \pm 50^\circ + k\cdot 120^\circ$ where k is an integer. To obtain these angles, we need to consider all suitable integer values of k in the expressions $x = 50^\circ + k\cdot 120^\circ$ and $x = -50^\circ + k\cdot 120^\circ$. First consider $x = 50^\circ + k\cdot 120^\circ$.

k	-1	0	1	2	3	4	5
$x = 50^\circ + k\cdot 120^\circ$	-70°	50°	170°	290°	410°	530°	650°

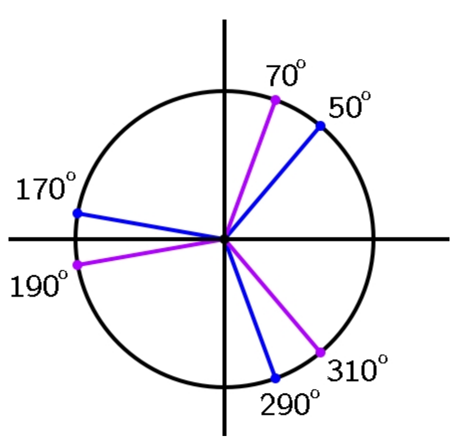
From all these values, 50°, 170°, and 290° fall between 0° and 360°. Similarly, we consider $x = -50^\circ + k\cdot 120^\circ$.

k	-1	0	1	2	3	4	5
$x = -50^\circ + k\cdot 120^\circ$	-170°	-50°	70°	190°	310°	430°	550°

From all these values, 70°, 190° and 310° fall between 0° and 360°. So the final answer is 50°, 70°, 170°, 190°, 290° and 310°.

c) Draw a picture of the solutions between 0° and 360°.

The group $x = 50^\circ + k\cdot 120^\circ$ (k integer) produces 50°, 170°, and 290° and the group $x = -50^\circ + k\cdot 120^\circ$ (k integer) produces 70°, 190° and 310°.



5. Consider the equation $\tan 5x = -1$.

a) Find all solutions for the equation. Present your answer in degrees.

Solution: We first solve for $5x$. Recall that the period of tangent is π and not 2π.

$$\begin{aligned} \tan 5x &= -1 \\ 5x &= -45^\circ + k \cdot 180^\circ \quad \text{where} \quad k \in \mathbb{Z} \qquad \text{divide by 5} \\ x &= -9^\circ + k \cdot 36^\circ \quad \text{where} \quad k \in \mathbb{Z} \end{aligned}$$

b) Find all solutions for the equation. Present your answer in radians.

Solution: We could just convert the answer from part a). Or, we can solve the equation in radians.

$$\begin{aligned} \tan 5x &= -1 \\ 5x &= -\frac{\pi}{4} + k\pi \quad \text{where } k \in \mathbb{Z} \qquad \text{divide by 5} \\ x &= -\frac{\pi}{20} + k\frac{\pi}{5} \qquad k \in \mathbb{Z} \end{aligned}$$

c) Find all solutions for the equation that fall between 0° and 360°.

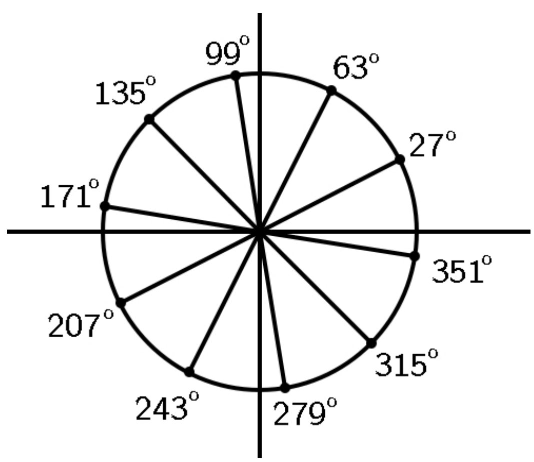
Solution: Substitute values into k starting with zero, and stopping once the solutions fall beyond 360°.

k	0	1	2	3	4	5	6	7	8	9	10	11
x (rad)	$-\frac{\pi}{20}$	$\frac{3\pi}{20}$	$\frac{7\pi}{20}$	$\frac{11\pi}{20}$	$\frac{15\pi}{20}$	$\frac{19\pi}{20}$	$\frac{23\pi}{20}$	$\frac{27\pi}{20}$	$\frac{31\pi}{20}$	$\frac{35\pi}{20}$	$\frac{39\pi}{20}$	$\frac{43\pi}{20}$
x (deg)	-9°	27°	63°	99°	135°	171°	207°	243°	279°	315°	351°	387°

So the solutions are: 27°, 63°, 99°, 135°, 171°, 207°, 243°, 279°, 315°, and 351°.

d) Draw a picture of the solutions between 0° and 360°.

Solution:



6. a) Solve the equation $-\sin 5x = \cos 10x$

b) List all solutions (in degrees) that fall between 0° and 360°.

Solution: Let us notice that 10 is twice 5 and so the double angle formula for cosine might be used. Let us denote $5x$ by B.

$$-\sin B = \cos 2B$$

We will use the double-angle formula for cosine. This formula has three forms, we will use the one that expresses things in terms of sine.

$$-\sin B = 1 - 2\sin^2 B$$

The equation is quadratic in $\sin B$. We solve for $\sin B$.

$$\begin{aligned} 2\sin^2 B - \sin B - 1 &= 0 \\ (2\sin B + 1)(\sin B - 1) &= 0 \end{aligned}$$

$$\sin B = -\frac{1}{2} \quad \text{or} \quad \sin B = 1$$

We now solve for B.

$$\sin B = -\frac{1}{2} \qquad \text{or} \qquad \sin B = 1$$

$$\begin{aligned} B &= -30^\circ + k \cdot 360^\circ && \text{or} && B = -90^\circ + k \cdot 360^\circ \quad \text{where } k \in \mathbb{Z} \\ B &= -150^\circ + k \cdot 360^\circ \end{aligned}$$

Recall that $B = 5x$.

$$\begin{aligned} 5x &= -30^\circ + k \cdot 360^\circ \quad \text{or} && 5x = -90^\circ + k \cdot 360^\circ \quad \text{where } k \in \mathbb{Z} \\ 5x &= -150^\circ + k \cdot 360^\circ \end{aligned}$$

We solve for x by dividing both sides by 5.

$$\begin{aligned} x &= -6^\circ + k \cdot 72^\circ \quad \text{or} && x = -18^\circ + k \cdot 72^\circ \quad \text{where } k \in \mathbb{Z} \\ x &= -25^\circ + k \cdot 72^\circ \end{aligned}$$

b) List all solutions (in degrees) that fall between 0° and 360°.

Solution: we start with the expression $-6^\circ + k \cdot 72^\circ$ (where $k \in \mathbb{Z}$) and substitute $k = 0, 1, 2, 3$, and 4. We obtain the angles

$$-6^\circ, 66^\circ, 138^\circ, 210^\circ, 282^\circ$$

Since -6° does not belong into the desired interval (between 0° and 360°), we need to replace that with a co-terminal angle that does. We can either add 360° to -6° or use $k = 5$ in the expression $-6^\circ + k \cdot 72^\circ$. Either way, we obtain 354° and so the list is

$$66^\circ, 138^\circ, 210^\circ, 282^\circ, 354^\circ$$

We apply the same method to the expressions $-25^\circ + k \cdot 72^\circ$ and obtain

$$-25^\circ, 47^\circ, 119^\circ, 191^\circ, 263^\circ \quad \text{and replace } -25^\circ \text{ with } 335^\circ$$

we apply the same method to the expression $-18^\circ + k \cdot 72^\circ$ and obtain

$$-18^\circ, 54^\circ, 126^\circ, 198^\circ, 270^\circ \quad \text{and replace } -18^\circ \text{ with } 342^\circ$$

So the complete list of all solutions between 0° and 360° is

$$47^\circ, 54^\circ, 66^\circ, 119^\circ, 126^\circ, 138^\circ, 191^\circ, 198^\circ, 210^\circ, 263^\circ, 270^\circ, 282^\circ, 335^\circ, 342^\circ, 354^\circ$$

For more documents like this, visit our page at https://teaching.martahidegkuti.com and click on Lecture Notes. E-mail questions or comments to mhidegkuti@ccc.edu.