

Sample Problems

1. Suppose that θ is an acute angle, i.e. $0 < \theta < 90^\circ$. Compute the exact value of $\cos \theta$ and $\tan \theta$, given that $\sin \theta = \frac{2}{5}$.
2. Suppose that β is an acute angle, i.e. $0 < \beta < 90^\circ$. Compute the exact value of all trigonometric function values of β , given that $\tan \beta = 3$.
3. Suppose that α is an acute angle, i.e. $0 < \alpha < 90^\circ$. Compute the exact value of all trigonometric values of α , given that $\sin \alpha = x$.
4. Suppose that θ is an acute angle, i.e. $0 < \theta < 90^\circ$. Compute the exact value of all trigonometric values of θ , given that $\tan \theta = x$.

Practice Problems

1. Compute the exact value of all trigonometric functions of α if α is an acute angle with $\cos \alpha = \frac{3}{7}$.
Rationalize the denominator in the answer.
2. Compute the exact value of all trigonometric functions of β if β is an acute angle with $\csc \beta = 4$.
Rationalize the denominator in the answer.
3. Compute the exact value of all trigonometric functions of γ if γ is an acute angle with $\sec \gamma = \frac{2}{3}$.
Rationalize the denominator in the answer.
4. Compute the exact value of all trigonometric functions of α if α is an acute angle with $\cot \alpha = \frac{3}{7}$.
Rationalize the denominator in the answer.
5. Compute the exact value of all trigonometric functions of β if β is an acute angle with $\sin \beta = x$.
6. Compute the exact value of all trigonometric functions of α if α is an acute angle with $\tan \alpha = x$.
7. Compute the exact value of all trigonometric functions of θ if θ is an acute angle with $\sec \theta = \frac{5}{2}$.

Answers - Sample Problems

$$1. \cos \theta = \frac{\sqrt{21}}{5} \text{ and } \tan \theta = \frac{2}{\sqrt{21}} = \frac{2\sqrt{21}}{21}$$

$$2. \sin \beta = \frac{3}{\sqrt{10}} = \frac{3\sqrt{10}}{10}, \cos \beta = \frac{1}{\sqrt{10}} = \frac{\sqrt{10}}{10}, \text{ and } \tan \beta = 3, \csc \beta = \frac{\sqrt{10}}{3}, \sec \beta = \frac{\sqrt{10}}{1}, \text{ and } \cot \beta = \frac{1}{3}$$

$$3. \sin \alpha = x, \cos \alpha = \sqrt{1-x^2}, \tan \alpha = \frac{x}{\sqrt{1-x^2}}, \csc \alpha = \frac{1}{x}, \sec \alpha = \frac{1}{\sqrt{1-x^2}}, \text{ and } \cot \alpha = \frac{\sqrt{1-x^2}}{x}$$

$$4. \sin \theta = \frac{x}{\sqrt{x^2+1}}, \cos \theta = \frac{1}{\sqrt{x^2+1}}, \tan \theta = x, \csc \theta = \frac{\sqrt{x^2+1}}{x}, \sec \theta = \sqrt{x^2+1}, \cot \theta = \frac{1}{x}$$

Practice Problems - Answers

$$1. \sin \alpha = \frac{2\sqrt{10}}{7}, \cos \alpha = \frac{3}{7}, \tan \alpha = \frac{2\sqrt{10}}{3}, \csc \alpha = \frac{7\sqrt{10}}{20}, \sec \alpha = \frac{7}{3}, \cot \alpha = \frac{3\sqrt{10}}{20}$$

$$2. \sin \beta = \frac{1}{4}, \cos \beta = \frac{\sqrt{15}}{4}, \tan \beta = \frac{\sqrt{15}}{15}, \csc \beta = 4, \sec \beta = \frac{4\sqrt{15}}{15}, \cot \beta = \sqrt{15}$$

$$3. \text{ This is impossible, } \sec \gamma \text{ must have a value greater than one. There is no angle with } \sec \gamma = \frac{2}{3}.$$

$$4. \sin \alpha = \frac{7\sqrt{58}}{58}, \cos \alpha = \frac{3\sqrt{58}}{58}, \tan \alpha = \frac{7}{3}, \csc \alpha = \frac{\sqrt{58}}{7}, \sec \alpha = \frac{\sqrt{58}}{3}, \cot \alpha = \frac{3}{7}$$

$$5. \sin \beta = x, \cos \beta = \sqrt{1-x^2}, \tan \beta = \frac{x}{\sqrt{1-x^2}}, \csc \beta = \frac{1}{x}, \sec \beta = \frac{1}{\sqrt{1-x^2}}, \cot \beta = \frac{\sqrt{1-x^2}}{x}$$

$$6. \sin \alpha = \frac{x}{\sqrt{x^2+1}}, \cos \alpha = \frac{1}{\sqrt{x^2+1}}, \tan \alpha = x, \csc \alpha = \frac{\sqrt{x^2+1}}{x}, \sec \alpha = \sqrt{x^2+1}, \cot \alpha = \frac{1}{x}$$

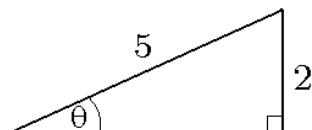
$$7. \sin \theta = \frac{\sqrt{21}}{5}, \cos \theta = \frac{2}{5}, \tan \theta = \frac{\sqrt{21}}{2}, \csc \theta = \frac{5\sqrt{21}}{21}, \sec \theta = \frac{5}{2}, \cot \theta = \frac{2\sqrt{21}}{21}$$

Sample Problems - Solutions

1. Suppose that θ is an acute angle, i.e. $0 < \theta < 90^\circ$. Compute the exact value of $\cos \theta$ and $\tan \theta$, given that $\sin \theta = \frac{2}{5}$.

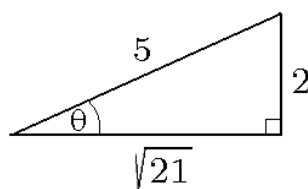
Solution 1 (Geometrical approach) It is given that $\sin \theta = \frac{2}{5}$. We draw a right triangle where this happens.

We draw a hypotenuse of length 5 units and a shorter side of length 2 units and draw in θ so that $\sin \theta = \frac{2}{5}$



is true in the triangle.

We now apply the Pythagorean theorem to find the length



of the missing side, $\sqrt{21}$.
values of θ from the picture.

We can now easily read all other trigonometric function

$$\cos \theta = \frac{\sqrt{21}}{5} \quad \text{and} \quad \tan \theta = \frac{2}{\sqrt{21}}$$

We finally rationalize the value for $\tan \theta$:

$$\tan \theta = \frac{2}{\sqrt{21}} \cdot \frac{\sqrt{21}}{\sqrt{21}} = \frac{2\sqrt{21}}{21}$$

Solution 2 (Algebraic approach) We start by the Pythagorean identity and solve for $\cos \theta$.

$$\begin{aligned} \sin^2 \theta + \cos^2 \theta &= 1 \\ \cos^2 \theta &= 1 - \sin^2 \theta \\ \cos \theta &= \pm \sqrt{1 - \sin^2 \theta} \end{aligned}$$

Since θ is an acute angle, all trigonometric values are positive and so we can discard the negative solution.

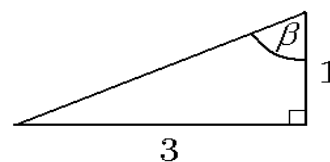
$$\cos \theta = \sqrt{1 - \sin^2 \theta} = \sqrt{1 - \left(\frac{2}{5}\right)^2} = \sqrt{1 - \frac{4}{25}} = \sqrt{\frac{25-4}{25}} = \sqrt{\frac{21}{25}} = \frac{\sqrt{21}}{\sqrt{25}} = \frac{\sqrt{21}}{5}$$

Now we can compute all other trigonometric values of θ

$$\tan \theta = \frac{\sin \theta}{\cos \theta} = \frac{\frac{2}{5}}{\frac{\sqrt{21}}{5}} = \frac{2}{5} \cdot \frac{5}{\sqrt{21}} = \frac{2}{\sqrt{21}} = \frac{2\sqrt{21}}{21}$$

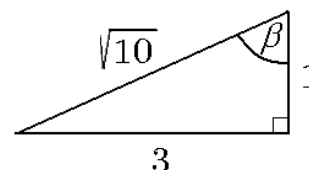
2. Suppose that β is an acute angle, i.e. $0 < \beta < 90^\circ$. Compute the exact value of all trigonometric function values of β , given that $\tan \beta = 3$.

Solution 1 (Geometrical approach) It is given that $\tan \beta = 3$. We will think of 3 as $\frac{3}{1}$ and draw a right triangle where $\tan \beta = \frac{3}{1}$ happens. That is a right triangle with shorter sides measuring 1 and 3 units, where the angle



opposite the longer side is β and so $\tan \beta = 3$ is true in the triangle.

We now



apply the Pythagorean theorem to find the hypotenuse. We obtain $\sqrt{10}$.

We can

now use the picture to read all other trigonometric function values of β . We also rationalize all values.

$$\begin{aligned}\sin \beta &= \frac{3}{\sqrt{10}} = \frac{3\sqrt{10}}{10}, & \cos \beta &= \frac{1}{\sqrt{10}} = \frac{\sqrt{10}}{10}, & \text{and} & \tan \beta = \frac{3}{1} = 3 \\ \csc \beta &= \frac{\sqrt{10}}{3}, & \sec \beta &= \frac{\sqrt{10}}{1}, & \text{and} & \cot \beta = \frac{1}{3}\end{aligned}$$

Solution 2 (Algebraic approach) We start by the Pythagorean identity and solve for $\cos \beta$.

$$\begin{aligned}\sin^2 \beta + \cos^2 \beta &= 1 && \text{divide by } \cos^2 \beta \\ \frac{\sin^2 \beta}{\cos^2 \beta} + \frac{\cos^2 \beta}{\cos^2 \beta} &= \frac{1}{\cos^2 \beta} \\ \tan^2 \beta + 1 &= \frac{1}{\cos^2 \beta}\end{aligned}$$

Recall that $\frac{1}{\cos x} = \sec x$. The identity we derived, $\tan^2 x + 1 = \sec^2 x$ will play a very important role in calculus. We will proceed to solve for $\cos \beta$.

$$\begin{aligned}\frac{1}{\cos^2 \beta} &= \tan^2 \beta + 1 && \text{take reciprocal of both sides} \\ \cos^2 \beta &= \frac{1}{\tan^2 \beta + 1} \\ \cos \beta &= \pm \sqrt{\frac{1}{\tan^2 \beta + 1}} = \pm \frac{1}{\sqrt{\tan^2 \beta + 1}}\end{aligned}$$

Since β is an acute angle, all trigonometric values are positive and so we can discard the negative solution.

$$\cos \beta = \sqrt{\frac{1}{\tan^2 \beta + 1}} = \sqrt{\frac{1}{3^2 + 1}} = \sqrt{\frac{1}{10}} = \frac{1}{\sqrt{10}} = \frac{\sqrt{10}}{10}$$

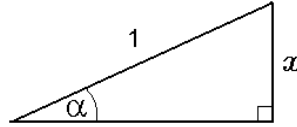
Now we can compute all other trigonometric values of β .

$$\begin{aligned}\tan \beta &= \frac{\sin \beta}{\cos \beta} && \text{multiply by } \cos \beta \\ \tan \beta \cos \beta &= \sin \beta \\ \sin \beta &= 3 \cdot \frac{\sqrt{10}}{10} = \frac{3\sqrt{10}}{10}\end{aligned}$$

$$\text{So } \sin \beta = \frac{3\sqrt{10}}{10}, \quad \cos \beta = \frac{\sqrt{10}}{10}, \quad \tan \beta = 3, \quad \csc \beta = \frac{\sqrt{10}}{3}, \quad \sec \beta = \sqrt{10}, \quad \text{and} \quad \cot \beta = \frac{1}{3}$$

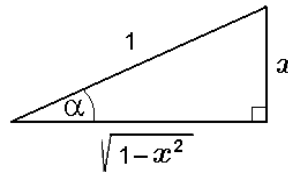
3. Suppose that α is an acute angle, i.e. $0 < \alpha < 90^\circ$. Compute the exact value of all trigonometric values of α , given that $\sin \alpha = x$.

Solution 1 (Geometrical approach) It is given that $\sin \alpha = x$. We can note right now that because x is the sine of an acute angle, x can not take just any value: it has to be between 0 and 1. We will think of x as $\frac{x}{1}$ and draw a right triangle where $\sin \alpha = \frac{x}{1}$ happens. We draw a right triangle and label the hypotenuse as 1.



We label one angle as α and the side opposite as x .

We now apply the Pythagorean



theorem to find the missing side. We obtain $\sqrt{1-x^2}$.
read all other trigonometric function values of α .

We can now use the picture to

$$\sin \alpha = \frac{x}{1} = x, \quad \cos \alpha = \sqrt{1-x^2}, \quad \tan \alpha = \frac{x}{\sqrt{1-x^2}}, \quad \csc \alpha = \frac{1}{x}, \quad \sec \alpha = \frac{1}{\sqrt{1-x^2}}, \quad \text{and} \quad \cot \alpha = \frac{\sqrt{1-x^2}}{x}$$

Solution 2 (Algebraic approach) We start by the Pythagorean identity and solve for $\cos \alpha$.

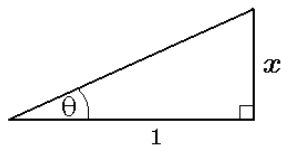
$$\begin{aligned} \sin^2 \alpha + \cos^2 \alpha &= 1 \\ \cos^2 \alpha &= 1 - \sin^2 \alpha \\ \cos \alpha &= \pm \sqrt{1 - \sin^2 \alpha} = \pm \sqrt{1 - x^2} \end{aligned}$$

Since α is an acute angle, all trigonometric values are positive and so we can discard the negative solution. So $\cos \alpha = \sqrt{1-x^2}$. Now we can compute all other trigonometric values of α .

$$\sin \alpha = x, \quad \cos \alpha = \sqrt{1-x^2}, \quad \tan \alpha = \frac{\sin \alpha}{\cos \alpha} = \frac{x}{\sqrt{1-x^2}}, \quad \csc \alpha = \frac{1}{x}, \quad \sec \alpha = \frac{1}{\sqrt{1-x^2}}, \quad \cot \alpha = \frac{\sqrt{1-x^2}}{x}$$

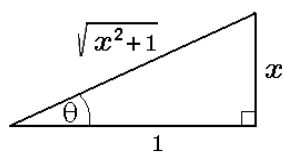
4. Suppose that θ is an acute angle, i.e. $0 < \theta < 90^\circ$. Compute the exact value of all trigonometric values of θ , given that $\tan \theta = x$.

Solution 1 (Geometrical approach) It is given that $\tan \theta = x$. We will think of x as $\frac{x}{1}$ and draw a right triangle where $\tan \theta = \frac{x}{1}$ happens. We draw a right triangle and label the shorter sides as x and 1. We label the



angle opposite x as θ .

We now apply the Pythagorean theorem to find the hypotenuse.



We obtain $\sqrt{x^2+1}$.

We can now use the picture to read all other trigonometric function

values of θ .

$$\sin \theta = \frac{x}{\sqrt{x^2 + 1}}, \quad \cos \theta = \frac{1}{\sqrt{x^2 + 1}}, \quad \tan \theta = \frac{x}{1} = x, \quad \csc \theta = \frac{\sqrt{x^2 + 1}}{x}, \quad \sec \theta = \sqrt{x^2 + 1}, \quad \text{and} \quad \cot \theta = \frac{1}{x}$$

Solution 2 (Algebraic approach) We start by the Pythagorean identity and solve for $\cos \theta$.

$$\begin{aligned} \sin^2 \theta + \cos^2 \theta &= 1 && \text{divide by } \cos^2 \theta \\ \frac{\sin^2 \theta}{\cos^2 \theta} + \frac{\cos^2 \theta}{\cos^2 \theta} &= \frac{1}{\cos^2 \theta} \\ \tan^2 \theta + 1 &= \frac{1}{\cos^2 \theta} && \text{take reciprocal of both sides} \\ \frac{1}{\tan^2 \theta + 1} &= \cos^2 \theta \\ \cos \theta &= \pm \sqrt{\frac{1}{\tan^2 \theta + 1}} = \pm \frac{1}{\sqrt{\tan^2 \theta + 1}} = \pm \frac{1}{\sqrt{x^2 + 1}} \end{aligned}$$

Since θ is an acute angle, all trigonometric values are positive and so we can discard the negative solution.

$$\cos \theta = \frac{1}{\sqrt{x^2 + 1}}$$

Now we can compute all other trigonometric values of θ .

$$\begin{aligned} \tan \theta &= \frac{\sin \theta}{\cos \theta} && \text{multiply by } \cos \theta \\ \tan \theta \cos \theta &= \sin \theta \\ \sin \theta &= \tan \theta \cos \theta = x \cdot \frac{1}{\sqrt{x^2 + 1}} = \frac{x}{\sqrt{x^2 + 1}} \end{aligned}$$

$$\sin \theta = \frac{x}{\sqrt{x^2 + 1}}, \quad \cos \theta = \frac{1}{\sqrt{x^2 + 1}}, \quad \tan \theta = x, \quad \csc \theta = \frac{\sqrt{x^2 + 1}}{x}, \quad \sec \theta = \sqrt{x^2 + 1}, \quad \cot \theta = \frac{1}{x}$$

For more documents like this, visit our page at <https://teaching.martahidegkuti.com> and click on Lecture Notes.
E-mail questions or comments to mhidegkuti@ccc.edu.