Sample Problems

Assume the following identities: For all x, y real numbers,

$$\boxed{\sin(x+y) = \sin x\cos y + \cos x\sin y \quad \text{and} \quad \cos(x+y) = \cos x\cos y - \sin x\sin y}$$

1. Find the formula for $\tan(x+y)$ in terms of $\tan x$ and $\tan y$.

2. Double-angle formulas.
   a) Find the formula for $\sin 2\alpha$.
   b) Find the formula for $\cos 2\alpha$
   c) Find all three forms of the double-angle formula for $\cos 2\alpha$.
   d) The double-angle formula for tangent.
      i) Use the identities for $\sin 2\alpha$ and $\cos 2\alpha$ to derive the formula for $\tan 2\alpha$.
      ii) Use the identity $\tan(x+y) = \dfrac{\tan x + \tan y}{1 - \tan x\tan y}$ to derive the formula for $\tan 2\alpha$.

3. Difference formulas.
   a) Use the unit circle to prove that for all x, $\sin(-x) = -\sin x$ and $\cos(-x) = \cos x$.
   b) Find and prove the formula for $\sin(x-y)$.
   c) Find and prove the formula for $\cos(x-y)$.
   d) The difference formula for tangent.
      i) Use the sum formula to derive the difference formula for tangent.
      ii) Use the identities for $\sin(x-y)$ and $\cos(x-y)$ to derive the formula for $\tan(x-y)$.

4. Which of the following is NOT equivalent to $\cos(x+270°)$?
   A) $\sin x$ B) $\cos(x-90°)$ C) $\cos(90°-x)$ D) $\sin(360°-x)$ E) $\sin(180°-x)$

5. If simplified, the expression $1 - \sin 2x\tan x$ is equivalent to which one of the following?
   A) $\cos 2x$ B) $\cos^2 x$ C) $\sin x\cos x$ D) $2\cos x$ E) $\sin^2 x - \cos^2 x$

6. Find the exact value for each of the following expressions.
   a) $\cos 75°$
   b) $\cos 68°\sin 8° - \sin 68°\cos 8°$
   c) $\dfrac{\tan\frac{2\pi}{15} + \tan\frac{\pi}{5}}{1 - \left(\tan\frac{2\pi}{15}\right)\left(\tan\frac{\pi}{5}\right)}$
   d) $\sin 80°\cos 50° - \cos 80°\sin 50°$
   e) $\sin 80°\cos 10° + \cos 80°\sin 10°$
   f) $\sin 195°$
   g) $\dfrac{2\tan\frac{5\pi}{12}}{1 - \tan^2\frac{5\pi}{12}}$
   h) $\tan 22.5°$

7. Solve each of the following equations. Present your answer as exact values, in radians.
   a) $2 + 3\sin x = \cos 2x$ b) $2\sin^2 x + \sin 2x = 0$

8. Suppose that $\sin\alpha = -\dfrac{8}{17}$ and α is not in the fourth quadrant; $\cos\beta = \dfrac{12}{13}$ and β is not in the first quadrant. Find the exact value for each of the following.
   a) $\tan(\alpha - \beta)$ b) $\cos(\alpha + \beta)$ c) $\sin 2\alpha$

9. Prove each of the following identities.
   a) $\cot 2x = \dfrac{\cot^2 x - 1}{2\cot x}$ b) $4\sin^4 x = 1 - 2\cos 2x + \cos^2 2x$ c) $\cos 3x = 4\cos^3 x - 3\cos x$

10. Find the exact value of $\tan\alpha$ if α is the acute angle formed by the lines $2x-3y=5$ and $5x+3y=1$.

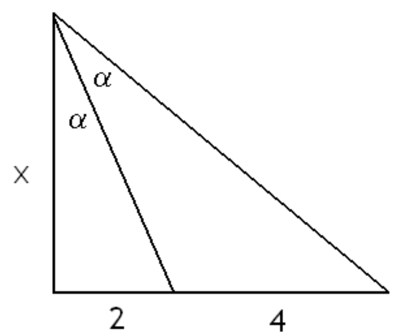
11. Compute $\tan\theta$ if we know that $\tan 2\theta = \dfrac{4}{3}$.

12. Let l be the line $y=\dfrac{3}{4}x$. Find an equation for the line that bisects the angle formed between l and the positive part of the $x-$axis.

13. Find $\sin\alpha$ if $\cos 2\alpha = \dfrac{7}{9}$ and α is in the first quadrant.

14. Find $\tan y$ if we know that $\tan x = 3$ and $\tan(x+y)=33$.

15. Find the exact values of x and α, based on the picture below.



Practice Problems

1. Find and prove the formula for $\cot(x+y)$.

2. Find the exact value for each of the following expressions.

a) $\sin 15°$

b) $\cos^2 15° - \sin^2 15°$

c) $\cos 255°$

d) $\dfrac{\tan 78° - \tan 18°}{1+\tan 78° \tan 18°}$

e) $\cos\left(\dfrac{25\pi}{12}\right)\sin\left(\dfrac{\pi}{3}\right) - \sin\left(\dfrac{25\pi}{12}\right)\cos\left(\dfrac{\pi}{3}\right)$

f) $\sin 22.5°$

g) $\cos 48° \cos 3° + \sin 48° \sin 3°$

3. Prove each of the following identities.

a) $\dfrac{1-\sin 2x}{\cos 2x} = \dfrac{1-\tan x}{1+\tan x}$

b) $\dfrac{1-\tan\frac{1}{2}x}{1+\tan\frac{1}{2}x} = \dfrac{1-\sin x}{\cos x}$

c) $\sin^2\dfrac{x}{2} = \dfrac{\csc x - \cot x}{2\csc x}$

d) $\sin 3x = 3\sin x - 4\sin^3 x$

e) $\tan x - \tan y = \dfrac{\sin(x-y)}{\cos x \cos y}$

f) $\cos 4t = 8\cos^4 t - 8\cos^2 t + 1$

g) $\sin\left(x+\dfrac{\pi}{6}\right) - \cos\left(x+\dfrac{\pi}{3}\right) = \sqrt{3}\sin x$

h) $\dfrac{1-\tan^2 x}{1+\tan^2 x} = \cos 2x$

4. Suppose that $\sin\alpha = -\frac{24}{25}$ and α is not in the fourth quadrant, and that $\cos\beta = \frac{5}{13}$ and β is not in the first quadrant. Find the **exact value** for each of the following.

   a) $\cos 2\alpha$ c) $\sin(\alpha+\beta)$ e) $\tan 2\alpha$ g) $\tan(\alpha-\beta)$

   b) $\sin 2\beta$ d) $\cos(\beta-\alpha)$ f) $\tan(\alpha+\beta)$ h) $\sin 3\beta$

5. Suppose that $\sin\alpha = \frac{3}{5}$ and α lies in quadrant I, and $\sin\beta = \frac{12}{13}$, and β lies in quadrant II. Find the exact values for each of the following

   a) $\sin 2\alpha$ c) $\sin(\alpha+\beta)$ e) $\tan(\alpha+\beta)$

   b) $\cos 2\alpha$ d) $\cos(\alpha-\beta)$ f) $\sin 3\alpha$

6. Find the exact value of $\tan\alpha$ if we know that $\tan\beta = \frac{1}{3}$ and $\tan(\alpha+\beta) = \frac{1}{2}$.

7. Solve each of the following equations.

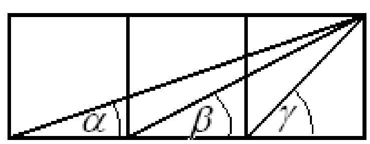
   a) $\cos 2x - \sin x = 1$ b) $\cos 2x + 3\sin x = 2$ c) $\sin x = \sin 2x$

8. Find the exact value of $\cos\alpha$ if $\cos 2\alpha = -\frac{119}{169}$ and α is in the second quadrant.

9. Let m be the straight line determined by the equation $2x - 3y = 8$ and k the line determined by $y = 6x - 15$. Find the exact value of the tangent of the angle formed by these lines.

10. Let l be the line $y = \frac{4}{3}x$. Find an equation for the line that bisects the angle formed between l and the positive part of the $x-$axis.

11. The figure below consists of three squares. Find the exact value of each of the following.



   a) $\tan\alpha$ b) $\tan\beta$ c) $\tan(\alpha+\beta)$ d) $\tan\gamma$

Sample Problems - Answers

1. $\tan(x+y) = \dfrac{\tan x + \tan y}{1 - \tan x \tan y}$

2. Double-angle formulas.

   a) $\sin 2\alpha = 2\sin\alpha\cos\alpha$ b) $\cos 2\alpha = \cos^2\alpha - \sin^2\alpha$

   c) $\cos 2\alpha = \cos^2\alpha - \sin^2\alpha$ $\cos 2\alpha = 1 - 2\sin^2\alpha$ $\cos 2\alpha = 2\cos^2\alpha - 1$

   d) $\tan 2\alpha = \dfrac{2\tan\alpha}{1-\tan^2\alpha}$

3. Difference formulas.
   a) see solutions
   b) $\sin(x-y) = \sin x \cos y - \cos x \sin y$
   c) $\cos(x-y) = \cos x \cos y + \sin x \sin y$
   d) $\tan(x-y) = \dfrac{\tan x - \tan y}{1 + \tan x \tan y}$
4. D
5. A
6. a) $\dfrac{\sqrt{6}-\sqrt{2}}{4}$ b) $-\dfrac{\sqrt{3}}{2}$ c) $\sqrt{3}$ d) $\dfrac{1}{2}$ e) 1 f) $\dfrac{\sqrt{2}-\sqrt{6}}{4}$ g) $-\dfrac{\sqrt{3}}{3}$ h) $\sqrt{2}-1$
7. a) $x = -\dfrac{\pi}{2} + 2k_1\pi, \quad -\dfrac{\pi}{6} + 2k_2\pi, \quad -\dfrac{5\pi}{6} + 2k_3\pi$ where $k_1, k_2, k_3 \in \mathbb{Z}$

   b) $x = k_1\pi, \quad x = -\dfrac{\pi}{4} + k_2\pi$ where $k_1, k_2 \in \mathbb{Z}$
8. a) $\dfrac{171}{140}$ b) $-\dfrac{220}{221}$ c) $-\dfrac{240}{289}$
9. see solutions
10. $\dfrac{1}{3}$
11. $\dfrac{1}{2}$ or -2
12. $y = \dfrac{1}{3}x$
13. $\dfrac{3}{10}$
14. 21
15. $\alpha = 30°, \quad x = 2\sqrt{3}$

Practice Problems - Answers

1. Find and prove the formula for $\cot(x+y)$.
   Solution: Use the sum-formulas for $\sin(x+y)$ and $\cos(x+y)$ and then divide both numerator and denominator by $\sin x \sin y$

$$\cot(x+y) = \frac{\cos(x+y)}{\sin(x+y)} = \frac{\cos x \cos y - \sin x \sin y}{\sin x \cos y + \cos x \sin y} = \frac{\dfrac{\cos x \cos y - \sin x \sin y}{\sin x \sin y}}{\dfrac{\sin x \cos y + \cos x \sin y}{\sin x \sin y}}$$

$$= \frac{\dfrac{\cos x \cos y}{\sin x \sin y} - \dfrac{\sin x \sin y}{\sin x \sin y}}{\dfrac{\sin x \cos y}{\sin x \sin y} + \dfrac{\sin x \sin y}{\sin x \sin y}} = \frac{\cot x \cot y - 1}{\cot y + \cot x}$$

2. a) $\dfrac{\sqrt{6}-\sqrt{2}}{4}$ b) $\dfrac{\sqrt{3}}{2}$ c) $\dfrac{\sqrt{2}-\sqrt{6}}{4}$ d) $\sqrt{3}$ e) $\dfrac{\sqrt{2}}{2}$ f) $\dfrac{1}{2}\sqrt{2-\sqrt{2}}$ g) $\dfrac{1}{\sqrt{2}}$

3. Prove each of the following identities.

a) $\dfrac{1-\sin 2x}{\cos 2x} = \dfrac{1-\tan x}{1+\tan x}$

Solution:

$$\begin{aligned} \text{RHS} &= \frac{1-\dfrac{\sin x}{\cos x}}{1+\dfrac{\sin x}{\cos x}} = \frac{\cos x - \sin x}{\cos x + \sin x} = \frac{\cos x - \sin x}{\cos x + \sin x}\cdot\frac{\cos x - \sin x}{\cos x - \sin x} = \frac{(\cos x - \sin x)^2}{\cos^2 x - \sin^2 x} \\ &= \frac{\cos^2 x + \sin^2 x - 2\sin x\cos x}{\cos 2x} = \text{LHS} \end{aligned}$$

b) $\dfrac{1-\tan\frac{1}{2}x}{1+\tan\frac{1}{2}x} = \dfrac{1-\sin x}{\cos x}$

Solution: Let $y = \dfrac{1}{2}x$ and then $2y = x$. Then we need to prove $\dfrac{1-\tan y}{1+\tan y} = \dfrac{1-\sin 2y}{\cos 2y}$ which we have already proven in part a).

c) $\sin^2\dfrac{x}{2} = \dfrac{\csc x - \cot x}{2\csc x}$

Solution: Write $y = \dfrac{x}{2}$. The we need to prove: $\sin^2 y = \dfrac{\csc 2y - \cot 2y}{2\csc 2y}$

$$\text{RHS} = \frac{\dfrac{1}{\sin 2y} - \dfrac{\cos 2y}{\sin 2y}}{2\dfrac{1}{\sin 2y}}$$

We multiply upstairs and downstairs by $\sin 2y$

$$= \frac{1-\cos 2y}{2} = \frac{1-\left(1-2\sin^2 y\right)}{2} = \frac{2\sin^2 y}{2} = \text{LHS}$$

d) $\sin 3x = 3\sin x - 4\sin^3 x$

Solution:

$$\begin{aligned} \text{LHS} &= \sin x\cos 2x + \cos x\sin 2x = \sin x\left(1-2\sin^2 x\right) + 2\sin x\cos^2 x \\ &= \sin x - 2\sin^3 x + 2\sin x\left(1-\sin^2 x\right) = \text{RHS} \end{aligned}$$

e) $\tan x - \tan y = \dfrac{\sin(x-y)}{\cos x\cos y}$

Solution:

$$\text{LHS} = \frac{\sin x}{\cos x} - \frac{\sin y}{\cos y} = \frac{\sin x\cos y - \cos x\sin y}{\cos x\cos y} = \frac{\sin(x-y)}{\cos x\cos y} = \text{RHS}$$

f) $\cos 4t = 8\cos^4 t - 8\cos^2 t + 1$

Solution: Recall that $\cos 2x = 2\cos^2 x - 1$

$$\begin{aligned} \text{LHS} &= \cos 4t = 2\cos^2(2t) - 1 = 2\left(2\cos^2 t - 1\right)^2 - 1 = \\ &= 2\left(4\cos^4 t - 4\cos^2 t + 1\right) - 1 = 8\cos^4 t - 8\cos^2 t + 1 = \text{RHS} \end{aligned}$$

g) $\sin\left(x+\frac{\pi}{6}\right)-\cos\left(x+\frac{\pi}{3}\right)=\sqrt{3}\sin x$

$$\begin{aligned}\text{LHS} &= \sin\left(x+\frac{\pi}{6}\right)-\cos\left(x+\frac{\pi}{3}\right)\\ &= \sin x\cos\left(\frac{\pi}{6}\right)+\cos x\sin\left(\frac{\pi}{6}\right)-\left(\cos x\cos\left(\frac{\pi}{3}\right)-\sin x\sin\left(\frac{\pi}{3}\right)\right)\\ &= \sin x\left(\frac{\sqrt{3}}{2}\right)+\cos x\left(\frac{1}{2}\right)-\cos x\left(\frac{1}{2}\right)+\sin x\left(\frac{\sqrt{3}}{2}\right)\\ &= \left(\frac{\sqrt{3}}{2}+\frac{\sqrt{3}}{2}\right)\sin x=\sqrt{3}\sin x=\text{RHS}\end{aligned}$$

h) $\dfrac{1-\tan^2 x}{1+\tan^2 x}=\cos 2x$

$$\text{LHS}=\frac{1-\tan^2 x}{1+\tan^2 x}=\frac{1-\dfrac{\sin^2 x}{\cos^2 x}}{1+\dfrac{\sin^2 x}{\cos^2 x}}=\frac{\cos^2 x-\sin^2 x}{\cos^2 x+\sin^2 x}=\text{RHS}$$

4. a) $-\frac{527}{625}$ b) $-\frac{120}{169}$ c) $-\frac{36}{325}$ d) $\frac{253}{325}$ e) $-\frac{336}{527}$ f) $\frac{36}{323}$ g) $-\frac{204}{253}$ h) $\frac{828}{2197}$

5. a) $\frac{24}{25}$ b) $\frac{7}{25}$ c) $\frac{33}{65}$ d) $\frac{16}{65}$ e) $-\frac{33}{56}$ f) $\frac{117}{125}$

6. $\frac{1}{7}$

7. a) $x=k_1\pi$, $x=-\frac{\pi}{6}+2k_2\pi$, $x=-\frac{5\pi}{6}+2k_3\pi$, where $k_1,k_2,k_3\in\mathbb{Z}$

b) $x=\frac{\pi}{2}+2k_1\pi$, $x=\frac{\pi}{6}+2k_2\pi$, $x=\frac{5\pi}{6}+2k_3\pi$, where $k_1,k_2,k_3\in\mathbb{Z}$

c) $x=k_1\pi$, $x=\pm\frac{\pi}{3}+2k_2\pi$ where $k_1,k_2\in\mathbb{Z}$

8. $-\frac{5}{13}$

9. $\frac{16}{15}$

10. $y=\frac{1}{2}x$

11. a) $\frac{1}{3}$ b) $\frac{1}{2}$ c) 1 d) 1

Sample Problems - Solutions

Assume the following identities: For all x, y real numbers,

$$\boxed{\sin(x+y) = \sin x \cos y + \cos x \sin y \quad \text{and} \quad \cos(x+y) = \cos x \cos y - \sin x \sin y}$$

1. Find the formula for $\tan(x+y)$ in terms of $\tan x$ and $\tan y$.
   Solution:
   $$\tan(x+y) = \frac{\sin(x+y)}{\cos(x+y)} = \frac{\sin x \cos y + \cos x \sin y}{\cos x \cos y - \sin x \sin y}$$
   We will now divide both numerator and denominator by $\cos x \cos y$
   $$\tan(x+y) = \frac{\dfrac{\sin x \cos y + \cos x \sin y}{\cos x \cos y}}{\dfrac{\cos x \cos y - \sin x \sin y}{\cos x \cos y}} = \frac{\dfrac{\sin x \cos y}{\cos x \cos y} + \dfrac{\cos x \sin y}{\cos x \cos y}}{\dfrac{\cos x \cos y}{\cos x \cos y} - \dfrac{\sin x \sin y}{\cos x \cos y}} = \frac{\dfrac{\sin x}{\cos x} + \dfrac{\sin y}{\cos y}}{1 - \dfrac{\sin x}{\cos x}\dfrac{\sin y}{\cos y}}$$
   $$= \frac{\tan x + \tan y}{1 - \tan x \tan y}$$

2. Double-angle formulas.

   a) Find the formula for $\sin 2\alpha$.
   Solution: Let $x = y = \alpha$. Then
   $$\sin(x+y) = \sin(\alpha+\alpha) = \sin\alpha\cos\alpha + \cos\alpha\sin\alpha = 2\sin\alpha\cos\alpha$$

   b) Find the formula for $\cos 2\alpha$.
   Solution: Let $x = y = \alpha$. Then
   $$\cos(x+y) = \cos(\alpha+\alpha) = \cos\alpha\cos\alpha - \sin\alpha\sin\alpha = \cos^2\alpha - \sin^2\alpha$$

   c) Find all three forms of the double-angle formula for $\cos 2\alpha$.
   Solution: We have $\cos 2\alpha = \cos^2\alpha - \sin^2\alpha$. We can eliminate $\cos\alpha$ using the identity $\cos^2\alpha + \sin^2\alpha = 1$; by re-writing $\cos^2\alpha = 1 - \sin^2\alpha$
   $$\cos 2\alpha = \cos^2\alpha - \sin^2\alpha = \left(1 - \sin^2\alpha\right) - \sin^2\alpha = 1 - 2\sin^2\alpha$$
   Similarly, we can eliminate $\sin\alpha$ using the identity $\cos^2\alpha + \sin^2\alpha = 1$; by re-writing $\sin^2\alpha = 1 - \cos^2\alpha$
   $$\cos 2\alpha = \cos^2\alpha - \sin^2\alpha = \cos^2\alpha - \left(1 - \cos^2\alpha\right) = 2\cos^2\alpha - 1$$

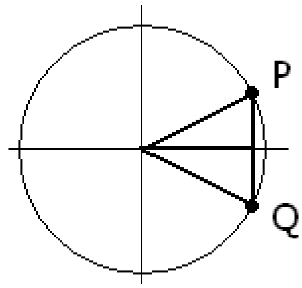
   d) The double-angle formula for tangent.

   i) Use the identities for $\sin 2\alpha$ and $\cos 2\alpha$ to derive the formula for $\tan 2\alpha$.
   Solution:
   $$\tan 2\alpha = \frac{\sin 2\alpha}{\cos 2\alpha} = \frac{2\sin\alpha\cos\alpha}{\cos^2\alpha - \sin^2\alpha}$$
   We will now divide both numerator and denominator by $\cos^2\alpha$.
   $$\tan 2\alpha = \frac{2\sin\alpha\cos\alpha}{\cos^2\alpha - \sin^2\alpha} = \frac{\dfrac{2\sin\alpha\cos\alpha}{\cos^2\alpha}}{\dfrac{\cos^2\alpha - \sin^2\alpha}{\cos^2\alpha}} = \frac{\dfrac{2\sin\alpha}{\cos\alpha}}{\dfrac{\cos^2\alpha}{\cos^2\alpha} - \dfrac{\sin^2\alpha}{\cos^2\alpha}} = \frac{2\tan\alpha}{1 - \tan^2\alpha}$$

   ii) Use the identity $\tan(x+y) = \dfrac{\tan x + \tan y}{1 - \tan x \tan y}$ to derive the formula for $\tan 2\alpha$.
   Solution: Let $x = y = \alpha$
   $$\tan 2\alpha = \tan(\alpha+\alpha) = \frac{\tan\alpha + \tan\alpha}{1 - \tan\alpha\tan\alpha} = \frac{2\tan\alpha}{1 - \tan^2\alpha}$$

3. Difference formulas.

a) Use the unit circle to prove that for all x, $\sin(-x) = -\sin x$ and $\cos(-x) = \cos x$.
Solution: When we draw an angle x (represented by point P) and its opposite (represented by point Q) we see that the two right triangles created are identical. We see that P and Q have the same first coordinate, thus $\cos(-x) = \cos x$ and that the second coordinates of p and Q are opposites, thus $\sin(-x) = -\sin x$.



b) Find and prove the formula for $\sin(x-y)$.
Solution: We will use the basic algebraic fact that to subtract is to add the opposite. We will think of $x-y$ as $x+(-y)$ and apply the sum formula to this sum.

$$\sin(x-y) = \sin(x+(-y)) = \sin x \cos(-y) + \cos x \sin(-y)$$

We know (see part a)) that $\sin(-y) = -\sin y$ and $\cos(-y) = \cos y$.

$$\sin(x-y) = \sin x \cos y + (-1)\cos x \sin y = \sin x \cos y - \cos x \sin y$$

c) Find and prove the formula for $\cos(x-y)$.
Solution: We will use the basic algebraic fact that to subtract is to add the opposite. We will think of $x-y$ as $x+(-y)$ and apply the sum formula to this sum.

$$\cos(x-y) = \cos(x+(-y)) = \cos x \cos(-y) - \sin x \sin(-y)$$

We know (see part a)) that $\sin(-y) = -\sin y$ and $\cos(-y) = \cos y$.

$$\cos(x-y) = \cos x \cos y - (-1)\sin x \sin y = \cos x \cos y + \sin x \sin y$$

d) The difference formula for tangent.

i) Use the sum formula to derive the difference formula for tangent.
Solution: First, we will prove that $\tan(-x) = -\tan x$ for all real numbers x.

$$\tan(-x) = \frac{\sin(-x)}{\cos(-x)} = \frac{-\sin x}{\cos x} = -\tan x$$

We will use the basic algebraic fact that to subtract is to add the opposite. We will think of $x-y$ as $x+(-y)$ and apply the sum formula to this sum.

$$\tan(x-y) = \tan(x+(-y)) = \frac{\tan x + \tan(-y)}{1 - \tan x \tan(-y)} = \frac{\tan x + (-1)\tan y}{1-(-1)\tan x \tan y} = \frac{\tan x - \tan y}{1 + \tan x \tan y}$$

ii) Use the identities for $\sin(x-y)$ and $\cos(x-y)$ to derive the formula for $\tan(x-y)$.
Solution:

$$\tan(x-y) = \frac{\sin(x-y)}{\cos(x-y)} = \frac{\sin x \cos y - \cos x \sin y}{\cos x \cos y + \sin x \sin y}$$

We will now divide both numerator and denominator by $\cos x \cos y$.

$$\tan(x-y) = \frac{\sin x \cos y - \cos x \sin y}{\cos x \cos y + \sin x \sin y} = \frac{\dfrac{\sin x \cos y - \cos x \sin y}{\cos x \cos y}}{\dfrac{\cos x \cos y + \sin x \sin y}{\cos x \cos y}}$$

$$= \frac{\dfrac{\sin x \cos y}{\cos x \cos y} - \dfrac{\cos x \sin y}{\cos x \cos y}}{\dfrac{\cos x \cos y}{\cos x \cos y} + \dfrac{\sin x \sin y}{\cos x \cos y}} = \frac{\dfrac{\sin x}{\cos x} - \dfrac{\sin y}{\cos y}}{1 + \dfrac{\sin x}{\cos x}\dfrac{\sin y}{\cos y}} = \frac{\tan x - \tan y}{1 + \tan x \tan y}$$

4. Which of the following is NOT equivalent to $\cos(x+270°)$?
A) $\sin x$ B) $\cos(x-90°)$ C) $\cos(90°-x)$ D) $\sin(360°-x)$ E) $\sin(180°-x)$
Solution: Let us first simplify $\cos(x+270°)$, using the sum formula for cosines.

$$\cos(x+270°) = \cos x \cos 270° - \sin x \sin 270° = \cos x\,(0) - \sin x\,(-1) = \sin x$$

and so our expression is equivalent to A. We can similarly simplify all the other expressions using a difference formula.

$$\cos(x-90°) = \cos x \cos 90° + \sin x \sin 90° = \cos x\,(0) + \sin x\,(1) = \sin x$$

so B is also equivalent to $\sin x$. Similarly,

$$\cos(90°-x) = \cos 90° \cos x + \sin 90° \sin x = (0)\cos x + (1)\sin x = \sin x$$

and so C is also equivalent to $\sin x$.

$$\sin(360°-x) = \sin 360° \cos x - \cos 360° \sin x = (0)\cos x - (1)\sin x = -\sin x$$

which is not equivalent to our expression. Finally,

$$\sin(180°-x) = \sin 180° \cos x - \cos 180° \sin x = (0)\cos x - (-1)\sin x = \sin x$$

and so E is also equivalent to $\sin x$.

5. If simplifed, the expression $1 - \sin 2x \tan x$ is equivalent to which one of the following?
A) $\cos 2x$ B) $\cos^2 x$ C) $\sin x \cos x$ D) $2\cos x$ E) $\sin^2 x - \cos^2 x$
Solution:

$$1 - \sin 2x \tan x = 1 - (2\sin x \cos x)\frac{\sin x}{\cos x} = 1 - 2\sin^2 x = \cos 2x$$

6. Find the exact value for each of the following expressions.

a) $\cos 75° = \cos(45°+30°) = \cos 45° \cos 30° - \sin 45° \sin 30° = \dfrac{\sqrt{2}}{2}\cdot\dfrac{\sqrt{3}}{2} - \dfrac{\sqrt{2}}{2}\cdot\dfrac{1}{2} = \dfrac{\sqrt{6}-\sqrt{2}}{4}$

b) $\cos 68° \sin 8° - \sin 68° \cos 8° = \sin(8°-68°) = \sin(-60°) = -\sin 60° = -\dfrac{\sqrt{3}}{2}$

c) $\dfrac{\tan\dfrac{2\pi}{15} + \tan\dfrac{\pi}{5}}{1 - \left(\tan\dfrac{2\pi}{15}\right)\left(\tan\dfrac{\pi}{5}\right)} = \tan\left(\dfrac{2\pi}{15} + \dfrac{\pi}{5}\right) = \tan\left(\dfrac{2\pi}{15} + \dfrac{3\pi}{15}\right) = \tan\left(\dfrac{5\pi}{15}\right) = \tan\dfrac{\pi}{3} = \sqrt{3}$

d) $\sin 80° \cos 50° - \cos 80° \sin 50° = \sin(80°-50°) = \sin 30° = \dfrac{1}{2}$

e) $\sin 80° \cos 10° + \cos 80° \sin 10° =$
Solution: Since $80° + 10° = 90°$, $\sin 80° = \cos 10°$ and $\cos 80° = \sin 10°$, and so we can re-write the expression as

$$\sin 80° \cos 10° + \cos 80° \sin 10° = \sin 80° \sin 80° + \cos 80° \cos 80° = \sin^2 80° + \cos^2 80° = 1$$

f) $\sin 195^\circ =$

$\sin(150^\circ + 45^\circ) = \sin 150^\circ \cos 45^\circ + \cos 150^\circ \sin 45^\circ = \frac{1}{2}\left(\frac{\sqrt{2}}{2}\right) + \left(-\frac{\sqrt{3}}{2}\right)\left(\frac{\sqrt{2}}{2}\right) = \frac{\sqrt{2}-\sqrt{6}}{4}$

g) $\dfrac{2\tan\frac{5\pi}{12}}{1-\tan^2\frac{5\pi}{12}} = \tan\left(2\cdot\frac{5\pi}{12}\right) = \tan\frac{5\pi}{6} = -\frac{\sqrt{3}}{3}$

h) $\tan 22.5^\circ$

Solution: Consider the double-angle formula for tangent. and set $\alpha = 22.5^\circ$.

$$\begin{aligned} \tan 2\alpha &= \frac{2\tan\alpha}{1-\tan^2\alpha} \qquad\qquad \alpha = 22.5^\circ \\ \tan 45^\circ &= \frac{2\tan 22.5^\circ}{1-\tan^2 22.5^\circ} \\ 1 &= \frac{2\tan 22.5^\circ}{1-\tan^2 22.5^\circ} \end{aligned}$$

Now write $a = \tan 22.5^\circ$ and solve the quadratic equation for a.

$$\begin{aligned} 1 &= \frac{2a}{1-a^2} \quad \text{multiply by } 1-a^2 \\ 1-a^2 &= 2a \\ 0 &= a^2+2a-1 \\ 0 &= \underbrace{a^2+2a+1}-1-1 \end{aligned}$$

$$0 = (a+1)^2 - 2$$
$$0 = (a+1)^2 - \left(\sqrt{2}\right)^2$$
$$0 = \left(a+1+\sqrt{2}\right)\left(a+1-\sqrt{2}\right)$$
$$a_1 = -1-\sqrt{2} \qquad a_2 = -1+\sqrt{2}$$

Since 22.5° is in the first quadrant, its tangent is clearly positive. We can use this to easily rule out a_1. The answer is $\sqrt{2}-1$.

7. Solve each of the following equations. Present your answer as exact values, in radians.

a) $2 + 3\sin x = \cos 2x$

Solution: Substitute $\cos 2x = 1 - 2\sin^2 x$. Then the equation becomes quadratic in $\sin x$.

$$\begin{aligned} 2+3\sin x &= 1-2\sin^2 x \\ 2\sin^2 x + 3\sin x + 1 &= 0 \\ (\sin x + 1)(2\sin x + 1) &= 0 \end{aligned}$$

$$\sin x + 1 = 0 \quad \text{or} \quad 2\sin x + 1 = 0$$

giving us two sets of solutions:

$$\begin{aligned} \sin x + 1 &= 0 \\ \sin x &= -1 \\ x &= -\frac{\pi}{2} + 2k\pi \text{ where } k \text{ is an integer} \end{aligned}$$

$$\begin{aligned} \text{or} \quad 2\sin x + 1 &= 0 \\ \sin x &= -\frac{1}{2} \\ x &= -\frac{\pi}{6} + 2k\pi \text{ where } k \text{ is an integer and} \\ x &= -\frac{5\pi}{6} + 2k\pi \text{ where } k \text{ is an integer.} \end{aligned}$$

So the set of all solutions is

$$x=-\frac{\pi}{2}+2k_1\pi \quad \text{and} \quad x=-\frac{\pi}{6}+2k_2\pi \quad \text{and} \quad -\frac{5\pi}{6}+2k_3\pi \quad \text{where } k_1,k_2,k_3 \text{ are integers}$$

b) $2\sin^2 x+\sin 2x=0$ $\qquad k\pi,\ \ x=-\frac{\pi}{4}+k\pi$, where $k\in\mathbb{Z}$

Solution:

$$\begin{aligned} 2\sin^2 x+\sin 2x &= 0 && \sin 2x = 2\sin x\cos x \\ 2\sin^2 x+2\sin x\cos x &= 0 && \text{factor out } \sin x \\ 2\sin x\left(\sin x+\cos x\right) &= 0 \end{aligned}$$

$$\sin x=0 \qquad \text{or} \qquad \sin x+\cos x=0$$

$$\sin x=0 \quad \Longrightarrow \quad x=k\pi \quad k\in\mathbb{Z}$$

$$\begin{aligned} \sin x+\cos x &= 0 \\ \sin x &= -\cos x \end{aligned}$$

If $\cos x=0$, then $\sin x=\pm 1$ and so $\sin x=-\cos x$ can not be true. So there is no solution where $\cos x=0$. We can therefore assume that $\cos x\neq 0$, and then we may divide both sides by $\cos x$.

$$\begin{aligned} \sin x &= -\cos x && \text{divide by } \cos x\neq 0 \\ \frac{\sin x}{\cos x} &= -1 \\ \tan x &= -1 \quad \Longrightarrow \quad x=-\frac{\pi}{4}+k\pi \quad k\in\mathbb{Z} \end{aligned}$$

So the set of all solutions is

$$x=k_1\pi \quad \text{and} \quad x=-\frac{\pi}{4}+k_2\pi \quad \text{where } k_1,k_2 \text{ are integers}$$

8. Suppose that $\sin\alpha=-\frac{8}{17}$ and α is not in the fourth quadrant; $\cos\beta=\frac{12}{13}$ and β is not in the first quadrant. Find the exact value for each of the following.
Solution: The conditions given place α into the third quadrant and β into the fourth quadrant. These will determine the signs of the other trigonometric functions.
Since α is in the third quadrant, $\cos\alpha$ is negative. For the rest,

$$\cos\alpha=-\sqrt{1-\sin^2\alpha}=-\sqrt{1-\left(-\frac{8}{17}\right)^2}=-\frac{15}{17}$$

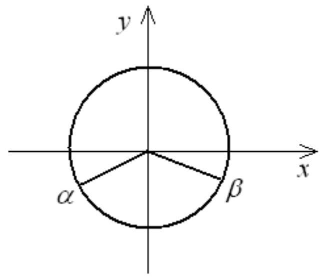
Similarly, since β is in the fourth quadrant, $\sin\beta$ is negative. For the rest,

$$\sin\beta=-\sqrt{1-\cos\beta^2}=-\sqrt{1-\left(\frac{12}{13}\right)^2}=-\frac{5}{13}$$

Now we have all we need:

$$\begin{aligned} \sin\alpha &= -\frac{8}{17} \qquad \cos\alpha=-\frac{15}{17} \qquad \tan\alpha=\frac{8}{15} \\ \sin\beta &= -\frac{5}{13} \qquad \cos\beta=\frac{12}{13} \qquad \tan\beta=-\frac{5}{12} \end{aligned}$$

Using the calculator, we can also come up with approximate values for α and β. These will give us a practical way to check our solution.



$$\alpha \cong 208.072^\circ \qquad \beta \cong -22.6199^\circ$$

a) $\tan(\alpha-\beta) = \dfrac{\tan\alpha - \tan\beta}{1+\tan\alpha\tan\beta} = \dfrac{\dfrac{8}{15} - \left(-\dfrac{5}{12}\right)}{1+\dfrac{8}{15}\left(-\dfrac{5}{12}\right)} = \dfrac{\dfrac{8(4)+5(5)}{60}}{1-\dfrac{2}{9}} = \dfrac{\dfrac{57}{60}}{\dfrac{7}{9}} = \dfrac{19}{20}\left(\dfrac{9}{7}\right) = \dfrac{171}{140}$

We can use the calculator to check our result by entering the fraction $\dfrac{171}{140} = 1.22143$ and $\tan(208.072^\circ - (-22.6199^\circ)) = 1.22143$ and compare the decimals.

b) $\cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta = \dfrac{-15}{17}\left(\dfrac{12}{13}\right) - \dfrac{-8}{17}\left(\dfrac{-5}{13}\right) = -\dfrac{220}{221}$

c) $\sin 2\alpha = 2\sin\alpha\cos\alpha = 2\left(-\dfrac{8}{17}\right)\left(\dfrac{15}{17}\right) = -\dfrac{240}{289}$

9. Prove each of the following identities.

a) $\cot 2x = \dfrac{\cot^2 x - 1}{2\cot x}$

$$\begin{aligned}\text{RHS} &= \frac{\cot^2 x - 1}{2\cot x} = \frac{\dfrac{\cos^2 x}{\sin^2 x} - 1}{2\dfrac{\cos x}{\sin x}} = \frac{\dfrac{\cos^2 x}{\sin^2 x} - \dfrac{\sin^2 x}{\sin^2 x}}{2\dfrac{\cos x}{\sin x}} = \frac{\dfrac{\cos^2 x - \sin^2 x}{\sin^2 x}}{\dfrac{2\cos x}{\sin x}} = \\ &= \frac{\cos^2 x - \sin^2 x}{\sin^2 x}\cdot\frac{\sin x}{2\cos x} = \frac{\cos^2 x - \sin^2 x}{2\sin x\cos x} = \frac{\cos 2x}{\sin 2x} = \cot 2x = \text{LHS}\end{aligned}$$

b) $4\sin^4 x = 1 - 2\cos 2x + \cos^2 2x$

$$\begin{aligned}\text{RHS} &= 1 - 2\cos 2x + \cos^2 2x = 1 - 2\left(1-2\sin^2 x\right) + \left(1-2\sin^2 x\right)^2 = \\ &= 1 - 2 + 4\sin^2 x + 1 - 4\sin^2 x + 4\sin^4 x = 4\sin^4 x = \text{LHS}\end{aligned}$$

c) $\cos 3x = 4\cos^3 x - 3\cos x$

$$\begin{aligned}\text{LHS} &= \cos 3x = \cos(x+2x) = \cos x\cos 2x - \sin x\sin 2x = \\ &= \cos x\left(2\cos^2 x - 1\right) - \sin x\,(2\sin x\cos x) = \\ &= 2\cos^3 x - \cos x - 2\sin^2 x\cos x = 2\cos^3 x - \cos x - 2\left(1-\cos^2 x\right)\cos x = \\ &= 2\cos^3 x - \cos x - 2\cos x\left(1-\cos^2 x\right) = 2\cos^3 x - \cos x - 2\cos x + 2\cos^3 x = \\ &= 4\cos^3 x - 3\cos x = \text{RHS}\end{aligned}$$

10. Find the exact value of $\tan\alpha$ if α is the acute angle formed by the lines $2x-3y=5$ and $5x+3y=1$.
Solution: Find the slopes of the lines first. We do this by solving the equations for y and then read the coefficients of x.

$$\begin{aligned} 2x-3y &= 5 \\ 2x-5 &= 3y \\ y &= \frac{2}{3}x-\frac{5}{3} \end{aligned} \qquad \text{and} \qquad \begin{aligned} 5x+3y=1 \\ 3y=-5x+1 \\ y=-\frac{5}{3}x+\frac{1}{3} \end{aligned}$$

So the slopes are $m_1=\tan\alpha=\dfrac{2}{3}$ and $m_2=\tan\beta=-\dfrac{5}{3}$. We are now looking for $\tan(\beta-\alpha)$.

$$\tan(\beta-\alpha)=\frac{\tan\beta-\tan\alpha}{1+\tan\beta\tan\alpha}=\frac{-\frac{5}{3}-\frac{2}{3}}{1+\left(-\frac{5}{3}\right)\frac{2}{3}}=\frac{-\frac{7}{3}}{1-\frac{10}{9}}=\frac{-\frac{7}{3}}{-\frac{1}{9}}=-\frac{7}{3}(-9)=21$$

Note: if we accidentally compute $\tan(\alpha-\beta)$, which is the tangent of the obtuse angle, the result would be -21, since the tangents of supplemental angles are opposites of each other. Then we would just have to conclude from that that the answer is 21.

11. Compute $\tan\theta$ if we know that $\tan 2\theta=\dfrac{4}{3}$.
Solution: Consider the double-angle formula for tangent.

$$\tan 2\theta=\frac{2\tan\theta}{1-\tan^2\theta}$$

If we know $\tan 2\theta$, then we can solve for $\tan\theta$. We will introduce a new variable $a=\tan\theta$. then $\tan 2\theta=\dfrac{2\tan\theta}{1-\tan^2\theta}$ becomes

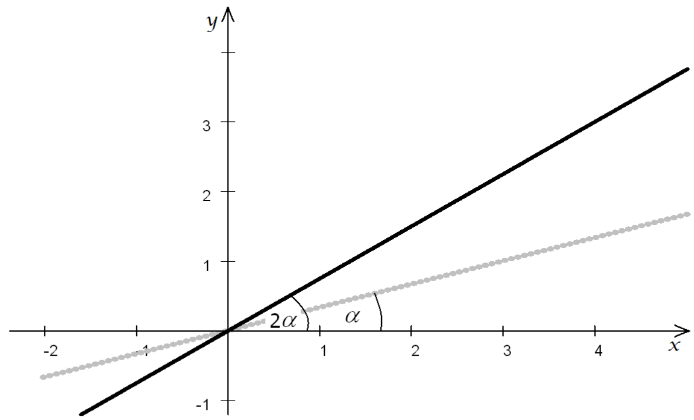
$$\begin{aligned} \frac{4}{3} &= \frac{2a}{1-a^2} && \text{multiply by } 3\left(1-a^2\right) \\ 4\left(1-a^2\right) &= 6a && \text{divide by 2} \\ 2\left(1-a^2\right) &= 3a \\ 2-2a^2 &= 3a \\ 0 &= 2a^2+3a-2 \\ 0 &= (2a-1)(a+2) \end{aligned}$$

$$a_1=\frac{1}{2} \text{ and } a_2=-2$$

Both solutions are correct. (Looking at the tangents $\frac{1}{2}$ and -2 as slopes we see that these two are perpendicular, and so the possible angles for θ differ by $90°$. After doubling, these values will differ by $180°$ and so they will have the same tangents. So, both are correct.)

12. Let l be the line $y = \frac{3}{4}x$. Find an equation for the line that bisects the angle formed between l and the positive part of the $x-$axis.

Solution: We sketch the problem first. Let α be exactly half of the angle formed between the line $y = \frac{3}{4}x$ and the positive part of the $x-$axis. Since both α and 2α are clearly acute angles, all trigonometric function values will be positive.



We need to find $\tan\alpha$ if $\tan 2\alpha = \frac{3}{4}$. Let us introduce the new variable $a = \tan\alpha$ and solve for a in the double angle formula for tangents.

$$\begin{aligned}
\tan 2\alpha &= \frac{2\tan\alpha}{1-\tan^2\alpha} \\
\frac{3}{4} &= \frac{2a}{1-a^2} \qquad \text{multiply by } 4\left(1-a^2\right) \\
3\left(1-a^2\right) &= 4(2a) \\
3-3a^2 &= 8a \\
0 &= 3a^2+8a-3 \\
0 &= (3a-1)(a+3) \qquad \Longrightarrow \quad a_1 = \frac{1}{3} \quad a_2 = -3
\end{aligned}$$

Since $a = \tan\alpha$ must be positive, we rule out -3 and so the answer is $\frac{1}{3}$. This is the slope of the line we are looking for. Since this line passes through the origin, its equation is $y = \frac{1}{3}x$.

13. Find $\sin\alpha$ if $\cos 2\alpha = \frac{7}{9}$ and α is in the first quadrant.

Solution: Solve the equation $\cos 2\alpha = 1 - 2\sin^2\alpha$ for $\sin\alpha$. Since α is in the first quadrant, $\sin\alpha$ is positive.

$$\begin{aligned}
\cos 2\alpha &= 1-2\sin^2\alpha \\
2\sin^2\alpha &= 1-\cos 2\alpha \\
\sin^2\alpha &= \frac{1-\cos 2\alpha}{2} \\
\sin\alpha &= \pm\sqrt{\frac{1-\cos 2\alpha}{2}} = \pm\sqrt{\frac{1-\frac{7}{9}}{2}} = \pm\frac{1}{3}
\end{aligned}$$

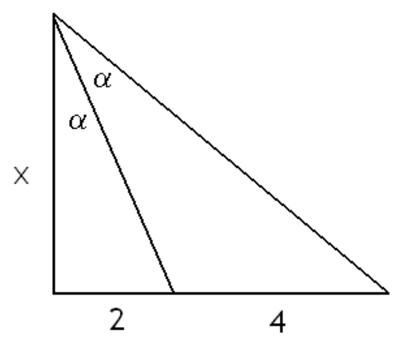
Since $\sin\alpha$ is positive, the answer is $\frac{1}{3}$.

14. Find $\tan y$ if we know that $\tan x = 3$ and $\tan(x+y) = 33$.

Solution: $\tan(x+y) = \dfrac{\tan x + \tan y}{1 - \tan x \tan y}$. We substitute $\tan x = 3$ and solve for $\tan y$.

$$\begin{aligned} \tan(x+y) &= \frac{\tan x + \tan y}{1 - \tan x \tan y} \\ 33 &= \frac{3 + \tan y}{1 - 3\tan y} \\ 33(1 - 3\tan y) &= 3 + \tan y \\ 33 - 99\tan y &= 3 + \tan y \\ 30 &= 100 \tan y \\ \frac{3}{10} &= \tan y \end{aligned}$$

15. Find the exact values of x and α, based on the picture below.



Solution: $\tan \alpha = \dfrac{2}{x}$ and $\tan 2\alpha = \dfrac{6}{x}$.

$$\begin{aligned} \tan 2\alpha &= \frac{2\tan\alpha}{1 - \tan^2\alpha} \\ \frac{6}{x} &= \frac{2\left(\frac{2}{x}\right)}{1 - \left(\frac{2}{x}\right)^2} && \text{solve for } x \\ \frac{6}{x} &= \frac{\frac{4}{x}}{1 - \frac{4}{x^2}} \cdot \frac{x^2}{x^2} && \text{divide by 2} \\ \frac{3}{x} &= \frac{2x}{x^2 - 4} && \text{multiply by } x\left(x^2 - 4\right) \\ 3\left(x^2 - 4\right) &= 2x^2 \\ 3x^2 - 12 &= 2x^2 \\ x^2 &= 12 \quad \Longrightarrow \quad x = \pm\sqrt{12} \end{aligned}$$

We rule out the negative root since x is a distance. Then $\tan\alpha = \dfrac{2}{x} = \dfrac{2}{\sqrt{12}} = \dfrac{2}{2\sqrt{3}} = \dfrac{1}{\sqrt{3}}$ thus $\alpha = 30°$.

