Now that we have defined trigonometric functions of acute angles, we will observe a rich structure of connections that immediately follow from the definitions.

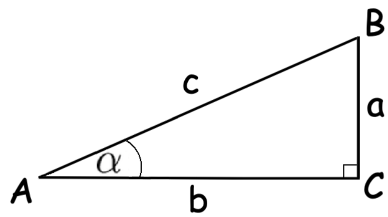
Let us recall that an identity is a type of an equation whose solution set is its entire domain. In other words, any number that can be substituted into both sides is also an equation. In this case, all acute angles will be solutions of the equations presented.

Theorem: The Pythagorean identity. For any acute angle α,

$$\sin^2\alpha + \cos^2\alpha = 1$$

This is perhaps the most fundamental trigonometric identity. As its name suggests, it has a lot to do with the Pythagorean theorem.

Consider a right triangle with the usual standard labeling of sides and angles.



$\sin\alpha = \dfrac{a}{c}$ and $\cos\alpha = \dfrac{b}{c}$

$$\begin{aligned}\text{Then}\quad \sin^2\alpha + \cos^2\alpha &= \left(\frac{a}{c}\right)^2 + \left(\frac{b}{c}\right)^2 = \frac{a^2}{c^2} + \frac{b^2}{c^2} \\ &= \frac{a^2+b^2}{c^2} = \frac{c^2}{c^2} = 1\end{aligned}$$

Theorem: For any acute angle α, $\dfrac{\sin\alpha}{\cos\alpha} = \tan\alpha$

Using the same labeling of sides and angles,

$$\frac{\sin\alpha}{\cos\alpha} = \frac{\frac{a}{c}}{\frac{b}{c}} = \frac{a}{c}\cdot\frac{c}{b} = \frac{a}{b} = \tan\alpha$$

A very important consequence of this simple identity is a version of the Pythagorean identity which we get of we start with $\sin^2\alpha + \cos^2\alpha = 1$ and divide both sides by $\cos^2\alpha$

$$\begin{aligned}\sin^2\alpha + \cos^2\alpha &= 1 \\ \frac{\sin^2\alpha}{\cos^2\alpha} + \frac{\cos^2\alpha}{\cos^2\alpha} &= \frac{1}{\cos^2\alpha} \\ \left(\frac{\sin\alpha}{\cos\alpha}\right)^2 + 1 &= \frac{1}{\cos^2\alpha} \\ \tan^2\alpha + 1 &= \sec^2\alpha\end{aligned}$$

This Pythagorean identity is going to be very useful in calculus.

Theorem: The Co-Function identities. If α and β are complements, i.e. $\alpha + \beta = 90°$,then

$\sin\alpha = \cos\beta$	$\csc\alpha = \sec\beta$
$\cos\alpha = \sin\beta$	$\sec\alpha = \csc\beta$
$\tan\alpha = \cot\beta$	$\cot\alpha = \tan\beta$

These identities hold because the following is true for every acute angle α: In a right triangle containing α, the third angle is the complement of α. Therefore, an angle and its complement always sit in the same right triangle, so they have the same trigonometric ratios. The only difference is that a side that is adjacent to α is opposite to β. For example, using our standard labels for sides and angles, the ratio $\frac{a}{c}$ is sine for α but cosine for β. In fact, the word cosine is short for 'the sine of the complement'. So cosine of α is the sine of its complement, sine β.

Example 1. Simplify each of the following expressions. Use exact values.

a) $\dfrac{\sin 12°}{\cos 78°}$ b) $\cos 25° \sin 65° + \cos^2 65°$ c) $\cos 35° \sin 65° - \cos^2 35°$

Solution: a) If we are instructed to use exact values, then either the angles must be algebraically accessible (i.e. $30°$, $45°$, and $60°$), or there must be some sort of cancellation. Otherwise, the expressions cannot be simplified using exact values.

By the co-function identities, $\cos 78° = \sin(90° - 78°) = \sin 12°$, and therefore we are simply dividing a quantity by itself.

$$\frac{\sin 12°}{\cos 78°} = \frac{\sin 12°}{\sin 12°} = 1$$

b) Let us notice that $25°$ and $65°$ are not famous angles, but they are complements, their sum is $90°$. Therefore, the sine of one is the cosine of the other. This will help us realize that we are looking at the Pythagorean identity. $\cos 25° = \sin 65°$, and so

$$\cos 25° \sin 65° + \cos^2 65° = \sin 65° \sin 65° + \cos^2 65° = \sin^2 65° + \cos^2 65° = 1$$

c) Again, the expression will become simpler if we apply the co-function identities. We first factor out $\cos 35°$ and then notice that $\sin 35° = \cos 35°$.

$$\cos 35° \sin 65° - \cos^2 35° = \cos 35° (\sin 65° - \cos 35°) = \cos 35° (\cos 35° - \cos 35°) = \cos 35° \cdot 0 = 0$$

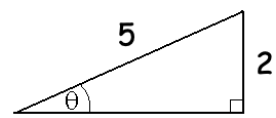
Part 2 - From One Trigonometric Expression to Another

The following techniques are also a straightforward consequence of the definitions of trigonometric functions in right triangles. If the angle is famous (i.e. $30°$, $45°$, and $60°$), we can always bounce back and forth between the angle and trigonometric ratios. This is not possible for the other angles, and so moving from angle to trigonometric ratios are a matter of approximation, using calculator or table of values.

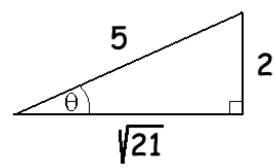
However, there is something we can always do, with any angle, and that is computing a trigonometric function when given another trigonometric function of the same angle. This is, as we will realize, just variations on the Pythagorean theorem. As easy this seems, it will be quite important later in trigonometry and also in calculus.

Example 2. Suppose that θ is an acute angle, i.e. $0 < \theta < 90°$. Compute the exact value of $\cos\theta$ and $\tan\theta$, given that $\sin\theta = \dfrac{2}{5}$.

Solution: It is given that $\sin\theta = \dfrac{2}{5}$. We draw a right triangle where this happens. We draw a hypotenuse of length 5 units and a shorter side of length 2 units and draw in θ so that $\sin\theta = \dfrac{2}{5}$ is true in the triangle.



We apply the Pythagorean theorem to find the length of the missing side that turns out to be $\sqrt{21}$.
We can now easily read all other trigonometric function values of θ from the picture.



$$\cos\theta = \frac{\sqrt{21}}{5} \quad \text{and} \quad \tan\theta = \frac{2}{\sqrt{21}}$$

We can rationalize the denominator of $\tan\theta$:

$$\tan\theta = \frac{2}{\sqrt{21}} \cdot \frac{\sqrt{21}}{\sqrt{21}} = \frac{2\sqrt{21}}{21}$$

There is another, purely algebraic way to solve this problem. We can use the Pythagorean identity, $\sin^2\theta + \cos^2\theta = 1$, and solve for $\cos\theta$.

$$\begin{aligned} \sin^2\theta + \cos^2\theta &= 1 \\ \cos^2\theta &= 1 - \sin^2\theta \\ \cos\theta &= \pm\sqrt{1 - \sin^2\theta} \end{aligned}$$

Since θ is an acute angle, all trigonometric values are positive and so we can discard the negative solution.

$$\cos\theta = \sqrt{1 - \sin^2\theta} = \sqrt{1 - \left(\frac{2}{5}\right)^2} = \sqrt{1 - \frac{4}{25}} = \sqrt{\frac{25-4}{25}} = \sqrt{\frac{21}{25}} = \frac{\sqrt{21}}{\sqrt{25}} = \frac{\sqrt{21}}{5}$$

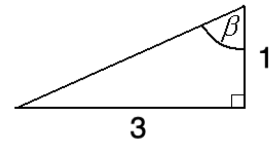
Now we can compute all other trigonometric values of θ, including the tangent: $\tan\theta = \dfrac{\sin\theta}{\cos\theta} = \dfrac{\frac{2}{5}}{\frac{\sqrt{21}}{5}} = \dfrac{2}{\sqrt{21}}$.

While the algebraic approach my be more elegant, it is notable that using the geometric approach, we get immediate access to all sic trigonometric functions of θ.

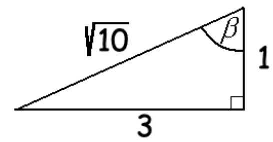
Example 3. Suppose that β is an acute angle, i.e. $0 < \beta < 90°$. Compute the exact value of all trigonometric function values of β, given that $\tan\beta = 3$.

Solution: The only difference between this example and the previous one is that now we have a single number, 3, while previously we had a fraction $\frac{2}{5}$. There is a very easy fix for that, we just need to think of 3 as $\frac{3}{1}$ and proceed as before.

We draw a right triangle where $\tan\beta = \frac{3}{1}$ happens. That is a right triangle with shorter sides measuring 1 and 3 units, where the angle opposite the longer side is β and so $\tan\beta = 3$ is true in the triangle.



We apply the Pythagorean theorem to find the length of the missing side that turns out to be $\sqrt{10}$. We can now easily read all other trigonometric function values of β from the picture.

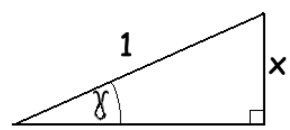


$$\sin\beta = \frac{3}{\sqrt{10}} \qquad \csc\beta = \frac{\sqrt{10}}{3}$$
$$\cos\beta = \frac{1}{\sqrt{10}} \qquad \sec\beta = \frac{\sqrt{10}}{1} = \sqrt{10}$$
$$\tan\beta = \frac{3}{1} = 3 \qquad \cot\beta = \frac{1}{3}$$

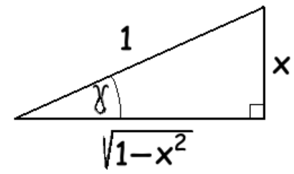
Example 4. Suppose that γ is an acute angle, i.e. $0 < \gamma < 90°$. Given that $\sin\gamma = x$, express all trigonometric function values of γ.

Solution: This problem is really the same as the one before, only a bit more abstract. All we need to know is that we can think of x as $\frac{x}{1}$. Notice that x can not be just any number. Because x is the sine of an angle, x cannot be just any number, $0 < x < 1$ must be true.

We draw a right triangle where $\sin\gamma = \frac{x}{1}$ happens.



We apply the Pythagorean theorem to find the length of the missing side. Of course we will not obtain numbers, just expressions in terms of x. Also note that the expression $\sqrt{1-x^2}$ cannot be simplified.



$$\sin\gamma = \frac{x}{1} = x \qquad \csc\gamma = \frac{1}{x}$$
$$\cos\gamma = \frac{\sqrt{1-x^2}}{1} = \sqrt{1-x^2} \qquad \sec\gamma = \frac{1}{\sqrt{1-x^2}}$$
$$\tan\gamma = \frac{x}{\sqrt{1-x^2}} \qquad \cot\gamma = \frac{\sqrt{1-x^2}}{x}$$

Practice Problems

1. Simplify each of the given expressions.

   a) $\dfrac{\sin 20^\circ + \sin 50^\circ}{\cos 70^\circ + \cos 40^\circ}$ b) $\sin 10^\circ \cos 80^\circ + \sin 10^\circ \cos 10^\circ \tan 80^\circ$ c) $\dfrac{\sin 13^\circ - \cos 77^\circ}{\sin 20^\circ + \cos 50^\circ}$

2. Compute the exact value of all trigonometric functions of α if α is an acute angle with $\cos \alpha = \dfrac{3}{7}$. Rationalize the denominator in the answer.

3. Compute the exact value of all trigonometric functions of β if β is an acute angle with $\csc \beta = 4$. Rationalize the denominator in the answer.

4. Compute the exact value of all trigonometric functions of γ if γ is an acute angle with $\sec \gamma = \dfrac{2}{3}$. Rationalize the denominator in the answer.

5. Compute the exact value of all trigonometric functions of α if α is an acute angle with $\cot \alpha = \dfrac{3}{7}$. Rationalize the denominator in the answer.

6. Compute the exact value of all trigonometric functions of β if β is an acute angle with $\sin \beta = x$.

7. Compute the exact value of all trigonometric functions of α if α is an acute angle with $\tan \alpha = x$.

8. Compute the exact value of all trigonometric functions of θ if θ is an acute angle with $\sec \theta = \dfrac{5}{2}$.

Answers

1. a) 1 b) 1 c) 0

2. $\sin\alpha = \dfrac{2\sqrt{10}}{7},\ \cos\alpha = \dfrac{3}{7},\ \tan\alpha = \dfrac{2\sqrt{10}}{3},\ \csc\alpha = \dfrac{7\sqrt{10}}{20},\ \sec\alpha = \dfrac{7}{3},\ \cot\alpha = \dfrac{3\sqrt{10}}{20}$

3. $\sin\beta = \dfrac{1}{4},\ \cos\beta = \dfrac{\sqrt{15}}{4},\ \tan\beta = \dfrac{\sqrt{15}}{15},\ \csc\beta = 4,\ \sec\beta = \dfrac{4\sqrt{15}}{15},\ \cot\beta = \sqrt{15}$

4. This is impossible, $\sec\gamma$ must have a value greater than one. There is no angle with $\sec\gamma = \dfrac{2}{3}$.

5. $\sin\alpha = \dfrac{7\sqrt{58}}{58},\ \cos\alpha = \dfrac{3\sqrt{58}}{58},\ \tan\alpha = \dfrac{7}{3},\ \csc\alpha = \dfrac{\sqrt{58}}{7},\ \sec\alpha = \dfrac{\sqrt{58}}{3},\ \cot\alpha = \dfrac{3}{7}$

6. $\sin\beta = x,\ \cos\beta = \sqrt{1-x^2},\ \tan\beta = \dfrac{x}{\sqrt{1-x^2}},\ \csc\beta = \dfrac{1}{x},\ \sec\beta = \dfrac{1}{\sqrt{1-x^2}},\ \cot\beta = \dfrac{\sqrt{1-x^2}}{x}$

7. $\sin\alpha = \dfrac{x}{\sqrt{x^2+1}},\ \cos\alpha = \dfrac{1}{\sqrt{x^2+1}},\ \tan\alpha = x,\ \csc\alpha = \dfrac{\sqrt{x^2+1}}{x},\ \sec\alpha = \sqrt{x^2+1},\ \cot\alpha = \dfrac{1}{x}$

8. $\sin\theta = \dfrac{\sqrt{21}}{5},\ \cos\theta = \dfrac{2}{5},\ \tan\theta = \dfrac{\sqrt{21}}{2},\ \csc\theta = \dfrac{5\sqrt{21}}{21},\ \sec\theta = \dfrac{5}{2},\ \cot\theta = \dfrac{2\sqrt{21}}{21}$