The law of sines can be used to compute angles and sides in triangles that may not have a right angle.

Theorem: (The Law of Sines) If a, b, and c are the three sides of a triangle with corresponding angles α, β, and γ, then
$$\frac{a}{\sin\alpha}=\frac{b}{\sin\beta}=\frac{c}{\sin\gamma}$$

Proof: Suppose that a, b, and c are the three sides of a triangle with corresponding angles α, β, and γ. Recall the theorem that the area of the triangle can be computed as $A=\frac{1}{2}ab\sin\gamma$. (The proof is on page 9.)
We can apply this theorem to the same triangle using different pairs of sides and the angle between them.

$$A=\frac{1}{2}ab\sin\gamma \quad \text{and} \quad A=\frac{1}{2}bc\sin\alpha \quad \text{and} \quad A=\frac{1}{2}ac\sin\beta$$

We computed the area of the same triangle, and so the area must be the same in each of the cases. Looking at the first and second equation, we get that

$$\begin{aligned}
\frac{1}{2}ab\sin\gamma &= \frac{1}{2}bc\sin\alpha && \text{multiply both sides by } 2\\
ab\sin\gamma &= bc\sin\alpha && \text{divide both sides by } b\\
a\sin\gamma &= c\sin\alpha && \text{divide both sides by } \sin\gamma\sin\alpha\\
\frac{a}{\sin\alpha} &= \frac{c}{\sin\gamma}
\end{aligned}$$

Looking at the first and third equation, we get that

$$\begin{aligned}
\frac{1}{2}ab\sin\gamma &= \frac{1}{2}ac\sin\beta && \text{multiply both sides by } 2\\
ab\sin\gamma &= ac\sin\beta && \text{divide both sides by } a\\
b\sin\gamma &= c\sin\beta && \text{divide both sides by } \sin\beta\sin\alpha\\
\frac{b}{\sin\beta} &= \frac{c}{\sin\gamma}
\end{aligned}$$

Thus we have proved that all three of $\frac{a}{\sin\alpha}$ and $\frac{b}{\sin\beta}$ and $\frac{c}{\sin\gamma}$ are equal.
The law of sines can be used to solve for sides or angles in triangles that may not be right triangles.

Definition: Given some sides and/or angles in a triangle, to **solve a triangle** means finding all missing sides and angles.

Example 1. Solve the triangle $c=3\text{ m}$, $\gamma=43°$, and $\alpha=81°$. Present approximate values of all answer, accurate up to three or four decimals.

Solution: We can easily find β since the other two angles were given.

$$\beta=180°-(\alpha+\gamma)=180°-(43°+81°)=180°-(43°+81°)=56°$$

We now know all three angles, which means that we know the triangle up to similarity. Since one side, namely c was also given, this uniquely determines the triangle. We will use the law of sines to find the missing sides, a and b.

$$\begin{aligned} \frac{a}{\sin\alpha} &= \frac{c}{\sin\gamma} \quad \text{solve for } a \quad a = \frac{c\sin\alpha}{\sin\gamma} = \frac{(3\text{ m})\sin 81^\circ}{\sin 43^\circ} \approx 4.34468\text{ m} \quad \text{and similarly,} \\ \frac{b}{\sin\beta} &= \frac{c}{\sin\gamma} \quad \text{solve for } b \quad b = \frac{c\sin\beta}{\sin\gamma} = \frac{(3\text{ m})\sin 56^\circ}{\sin 43^\circ} \approx 3.6468016\text{ m} \end{aligned}$$

Thus the solution is:

$$\begin{aligned} a &\approx 4.34468\text{ m}, \quad b \approx 3.6468016\text{ m}, \quad c = 3\text{ m} \\ \alpha &= 81^\circ, \qquad \beta = 56^\circ, \qquad \text{and} \quad \gamma = 37^\circ \end{aligned}$$

It is good practice to check whether the order between the sides is the same as the order between corresponding angles. Indeed, the shortest side is c and the smallest angle is γ, and the longest side is a and the greatest angle is α.

Part 2 - The Ambiguous Case

Consider the equation $\dfrac{\sin\alpha}{a} = \dfrac{\sin\beta}{b}$. This equation has four quantities in it. Generally speaking, we can solve for the fourth one given the other three. The missing quantity could be a side or an angle. In practical terms, there is a great difference between the two scenarios.

Suppose first that two angles and a side are given, and we must find the missing side. If two angles are given, we can easily find the third one, as the three angles in a triangle always add up to 180°. Therefore, all three angles are given. That uniquely determines the triangle up to similarity. There is just one triangle with three given angles, and we can shrink or enlarge it until the one given side matches the side of our triangle. In short, there is a unique solution if two angles and a side are given.

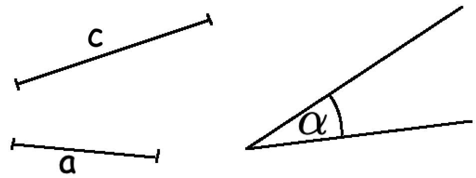
Looking at it algebraically, given side a and angles α and β, we can uniquely solve for side b.

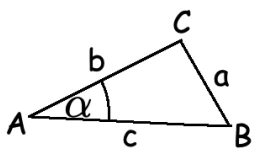
$$\frac{a}{\sin\alpha} = \frac{b}{\sin\beta} \quad \Rightarrow \quad b = \sin\beta \cdot \frac{a}{\sin\alpha}$$

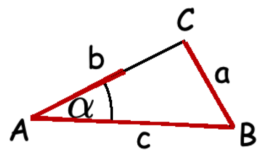
The other case is not that straightforward and it is often called **the ambiguous case**. Suppose that one angle and two sides are given, say sides a, b, and angle α. Using the equation $\dfrac{\sin\alpha}{a} = \dfrac{\sin\beta}{b}$, we can uniquely solve for $\sin\beta$ but that does not necessarily implies a unique β. For example, a unique solution $\sin\beta = \dfrac{1}{2}$ means that β could be 30° or 150°. Sometimes they both work and there are two different solutions for the ambiguous case. We will see examples for the four possible outcomes of the ambiguous solution.

Let us first consider the geometrical reasoning. There is a beautiful part of geometry called construction. Within that branch, to construct something means to precisely draw it using only two tools, a compass and a straight edge. The basics of construction can be found at https://www.mathsisfun.com/geometry/constructions.html

Suppose that a, b, and α were given, not as numbers, but as line segments and angle as drawn. How would we construct this triangle? It is always a good idea to draw the final product as well. There are two pictures: the final product and the final product with coloring of the sides and angle that were given.



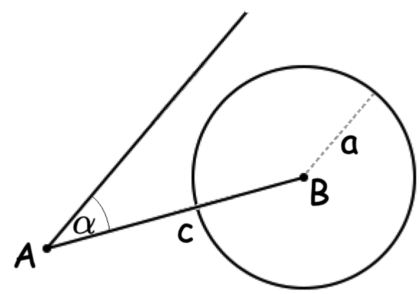




Let us start with line segment AB. Now all we need to find is the third vertex, C. At one end, point A, we measure the angle α. Point C must be on that ray. On the other side, at point B, C should be at a distance of a from B, but we don't know in what direction to draw the line segment. However, if we draw a circle centered at point B, with radius a, then point C must be on that circle. (Recall that a circle is the set of all points in the plane equidistant to a fixed point.) So where the ray starting at point A intersects the circle, that is point C. So finding point C is a matter of intersecting a circle and a line. There are four possible cases.

Case 1. There is no intersection point, and therefore no such triangle.
How do the numbers tell us that? Suppose that $a = 4$, $\alpha = 30°$, and $c = 10$. Then by the law of sines,

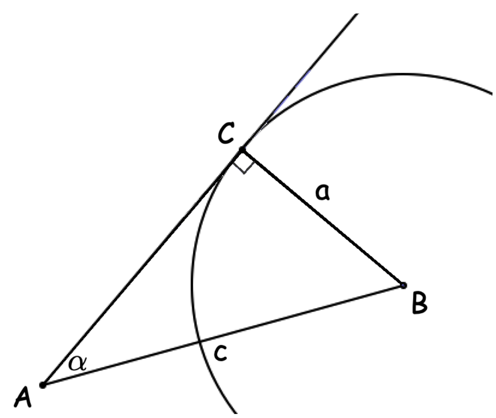
$$\frac{\sin\alpha}{a} = \frac{\sin\gamma}{c} \quad \Longrightarrow \quad \sin\gamma = c \cdot \frac{\sin\alpha}{a} = \frac{10 \cdot \sin 30°}{4} = \frac{10 \cdot \frac{1}{2}}{4} = \frac{5}{4} = 1.25$$



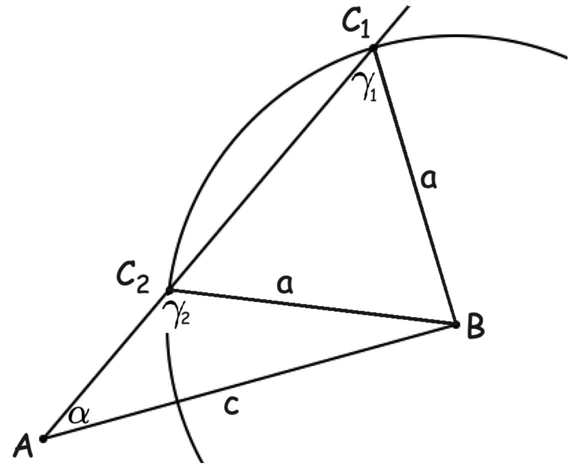
The equation $\sin\gamma = 1.25$ has no solutions as the sine of any angle is less than or equal to 1.

Case 2. There is one intersection point, the line is tangent to the circle.
How do the numbers tell us that? Suppose that $a = 5$, $\alpha = 30°$, and $c = 10$. Then by the law of sines,

$$\frac{\sin\alpha}{a} = \frac{\sin\gamma}{c} \quad \Longrightarrow \quad \sin\gamma = c \cdot \frac{\sin\alpha}{a} = \frac{10 \cdot \sin 30°}{5} = \frac{10 \cdot \frac{1}{2}}{5} = \frac{5}{5} = 1$$



The solution set of the equation $\sin\gamma = 1$ is $90° + k \cdot 360°$, where k is an integer. Among the infinitely many solutions, only one, $90°$ falls within $0°$ and $180°$. So there is one such triangle.



Case 3. There are two intersection points, and therefore two different solutions for the triangle.
How do the numbers tell us that? Suppose that $a = 8$, $\alpha = 30°$, and $c = 10$. Then by the law of sines,

$$\frac{\sin\alpha}{a} = \frac{\sin\gamma}{c} \quad\Longrightarrow\quad \sin\gamma = c\cdot\frac{\sin\alpha}{a} = \frac{10\cdot\sin 30°}{8} = \frac{10\cdot\frac{1}{2}}{8} = \frac{5}{8}$$

The equation $\sin\gamma = \frac{5}{8}$ has two solutions on the unit circle. If we enter $\sin^{-1}\left(\frac{5}{8}\right)$ into the calculator, it will display the value $38.6822°$ but its supplement has the same sine value and is also a solution. So $\gamma_1 = 38.6822°$ and $\gamma_2 = 180° - 38.6822° = 141.3178°$ and both angles will work.

To find the third angle, we add the two angles and subtract the sum from $180°$. Since we have two values for γ, we will perform each operation twice.

$$\beta_1 = 180° - (\alpha + \gamma_1) = 180° - (30° + 38.6822°) = 111.3178°$$

and

$$\beta_2 = 180° - (\alpha + \gamma_2) = 180° - (30° + 141.3178°) = 8.6822°$$

Now we will find the third side b - also twice.

$$\frac{b}{\sin\beta} = \frac{a}{\sin\alpha} \quad\Longrightarrow\quad b = \sin\beta\cdot\frac{a}{\sin\alpha}$$

$b_1 = \sin\beta_1\cdot\frac{a}{\sin\alpha} = \sin 111.3178°\cdot\frac{8}{\frac{1}{2}} = 14.9053$ units and $b_2 = \sin\beta_2\cdot\frac{a}{\sin\alpha} = \sin 8.6822°\cdot\frac{8}{\frac{1}{2}} = 2.4153$ units

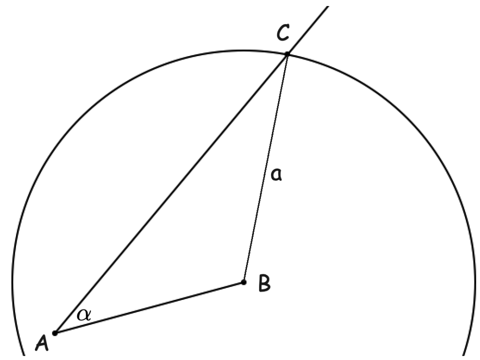
So there are two triangles:

$a_1 = 8,\ \ b_1 = 14.9053,\ \ c_1 = 10,\ \ \alpha_1 = 30°,\ \ \beta_1 = 111.3178°,\ \ \gamma_1 = 38.6822°$

and

$a_2 = 8,\ \ b_2 = 2.4153,\ \ c_2 = 10,\ \ \alpha_2 = 30°,\ \ \beta_2 = 8.6822°,\ \ \gamma_2 = 141.3178°$

Case 4. There is one intersection point, because the radius of the circle is too great for the second intersection.



How do the numbers tell us that? Suppose that $a = 16$, $\alpha = 30°$, and $c = 10$. Then by the law of sines,

$$\frac{\sin\alpha}{a} = \frac{\sin\gamma}{c} \quad \Longrightarrow \quad \sin\gamma = c \cdot \frac{\sin\alpha}{a} = \frac{10 \cdot \sin 30°}{16} = \frac{10 \cdot \frac{1}{2}}{16} = \frac{5}{16}$$

The solution set of the equation $\sin\gamma = \dfrac{5}{16}$ is $\gamma = 18.21° + k \cdot 360°$, and the supplements, $\gamma = 161.79° + k \cdot 360°$ where k is an integer. So it looks like there are two such triangles again.

Next we find the third angle by subtracting $\alpha + \gamma$ and subtract it from $180°$

$$\beta_1 = 180° - (\alpha + \gamma_1) = 180° - (30° + 18.21°) = 131.79°$$

and

$$\beta_2 = 180° - (\alpha + \gamma_2) = 180° - (30° + 161.79°) = 180° - 191.79° = -11.79°$$

and that's how the numbers are telling us that there is only one triangle. The two angles α and γ_2 add up to more than $180°$, so there is no room for the third angle.

We find the missing side as before, applying the law of sines.

$$b = \sin\beta \cdot \frac{a}{\sin\alpha} = \sin 131.79° \cdot \frac{16}{\frac{1}{2}} = 23.859 \text{ units}$$

So the only solution is $a = 16$, $b = 23.859$, $c = 10$, $\alpha = 30°$, $\beta = 131.79°$, $\gamma = 18.21°$

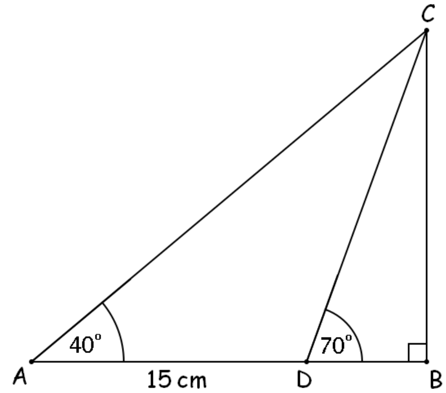
Is there a way to predict whether we will have case 3 or 4? Actually, there is. Recall that the order between sides corresponds to the measure of their opposite angles. The greatest angle is opposite the longest side, the shortest side is opposite the smallest angle, etc. Notice that in every triangle, only one angle can be obtuse (greater than $90°$) and must therefore be the greatest angle.

If we are dealing with the ambiguous case, two sides and one angle are given. The first thing we can compute is the angle opposite the side given, γ in our example. If the side corresponding to that angle, c in our case could be the longest side, there will be two solutions. If that side cannot be the longest side, there will be only one triangle. Given the two solutions for γ, one is acute and one is obtuse. If c cannot be the longest side, then γ cannot be the greatest angle and therefore it cannot be obtuse. Then only the acute value for γ will work.

Consider the previous examples. If $a = 8$, $\alpha = 30°$, and $c = 10$ are given, it is possible that c is the longest side and therefore γ could be the greatest angle, so it can be obtuse. There will be two solutions.

On the other hand, if $a = 16$, $\alpha = 30°$, and $c = 10$, then c cannot be the longest side because a is longer than c. Then γ cannot be the greatest angle and so it cannot be obtuse. In fact, it must be less than α. Therefore, there is only one solution.

Sample Problems



1. Prove that the area of a triangle can be computed as $A = \frac{1}{2}ab\sin\gamma$.

2. Prove the law of sines using results from problem 1.

3. Compute the length of line segment BD based on the picture given.

4. Prove that in any triangle, $P = \dfrac{a}{\sin\alpha}(\sin\alpha + \sin\beta + \sin\gamma)$.

5. Solve the given triangles. Present a decimal approximation of all answers, accurate up to four or more decimal places.

   a) $a = 3$, $\alpha = 42°$, $\beta = 100°$

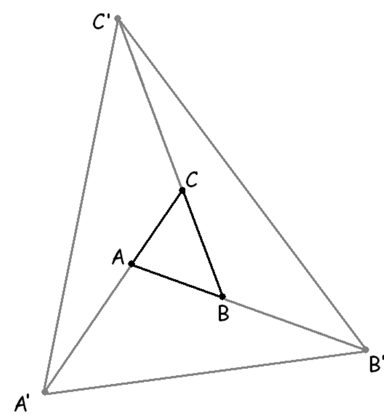
   b) $a = 10$, $b = 12$, $\alpha = 32°$

   c) $a = 7 \quad b = 9 \quad \alpha = 114°$

   d) $a = 22$ cm, $b = 23$ cm, $\alpha = 67°$

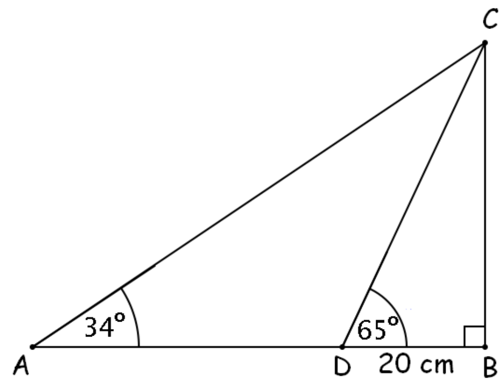
   e) $a = 22$ cm, $b = 23$ cm, $\beta = 67°$

   f) $a = 6,\ b = 12,\ \alpha = 30°$



6. Let ABC be any triangle. We extend the line segments AB, BC, and CA as shown on picture. Suppose that $\overline{AA'} = 2\overline{AC}$, and $\overline{BB'} = 2\overline{AB}$, and $\overline{CC'} = 2\overline{BC}$. What is the ratio of the area of triangle $A'B'C'$ to the area of triangle ABC? (Notation: AB denotes the line segment AB, while $\overline{AB}$ denotes **the length** of the line segment.)

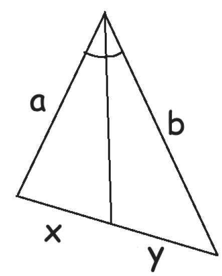
Practice Problems



1. Solve each of the following triangles.
   a) $\alpha = 106°$, $\beta = 21°$, $b = 2.4$ ft
   b) $\alpha = 62°$, $\gamma = 41°$, $a = 15$ cm
   c) $a = 12$ cm, $c = 17$ cm, $\gamma = 85°$
   d) $a = 12$ m, $b = 7$ m, $\beta = 65°$
   e) $a = 15$ ft, $c = 13$ ft, $\gamma = 27°$

2. Compute the length of line segment AC based on the picture given.

3. Use the formula $A = \frac{1}{2}ab\sin\gamma$ to find the area of an equilateral triangle of sides a.



4. Use the formula $A = \frac{1}{2}ab\sin\gamma$ to prove the angle bisector theorem. The angle bisector theorem states that the angle bisector of any angle in any triangle bisects the opposite side in such a way that the ratio of the two line segments is proportional to the ratio of the other two sides. In short, $\frac{a}{b} = \frac{x}{y}$.

Answers

Sample Problems

1. see solutions 2. see solutions 3. 6.59539 cm 4. see solutions

5. a) $\gamma = 38°$, $b \approx 4.41532$ unit, $c \approx 2.760275$ unit

   b) $\beta_1 = 39.487°$, $\gamma_1 = 108.513°$, $c_1 = 17.894$ and $\beta_2 = 140.513°$, $\gamma_2 = 7.487°$, $c_2 = 2.459$

   c) There is no triangle with the data given.

   d) $\beta_1 \approx 74.227°$, $\gamma_1 \approx 38.7730°$, $c_1 \approx 14.9670$ cm and $\beta_2 \approx 105.773°$, $\gamma_2 \approx 7.2270°$, $c_2 \approx 3.0066$ cm

   e) $\alpha \approx 61.70067°$, $\gamma \approx 51.29933°$, and $c \approx 19.499877$ cm.

   f) $\beta = 90°$, $\gamma = 60°$, and $c = 6\sqrt{3} \approx 10.392305$

6. 19 to 1

Practice Problems

1. a) $\gamma = 53°$, $a \approx 6.4376$ ft, $c \approx 5.348484$ ft b) $\beta = 77°$, $b \approx 16.55314$ cm, $c \approx 11.1455$ cm

   c) $\alpha \approx 44.684°$, $\beta \approx 50.316°$, $b \approx 13.1328$ cm d) no solution

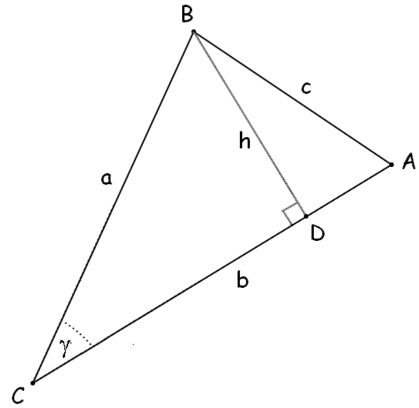
   e) $\alpha_1 = 31.43782584°$, $\beta_1 = 121.56217416°$, $b_1 \approx 24.399$ ft

   $\alpha_2 = 148.56217416°$, $\beta_2 = 4.43782584°$, $b_2 \approx 2.2157$ ft

2. 76.7 cm 3. $\dfrac{\sqrt{3}}{4}a^2$

Solutions — Sample Problems

1. Prove that the area of a triangle can be computed as $A = \dfrac{1}{2}ab\sin\gamma$.



Proof: Consider the picture shown. Let h denote the height belonging to side b. The area of the triangle can be computed as $\dfrac{1}{2}bh$. Consider now the right triangle BCD. In this triangle, $\sin\gamma = \dfrac{h}{a}$. We solve for h: $h = a\sin\gamma$.

$$A = \frac{1}{2}bh = \frac{1}{2}b\left(a\sin\gamma\right) = \frac{1}{2}ab\sin\gamma$$

2. Prove the law of sines using results from problem 1.

Proof: Consider triangle ABC. We compute the area of the triangle using the same formula but applying it to different sides.

$$A = \frac{1}{2}ab\sin\gamma = \frac{1}{2}ac\sin\beta = \frac{1}{2}bc\sin\alpha$$

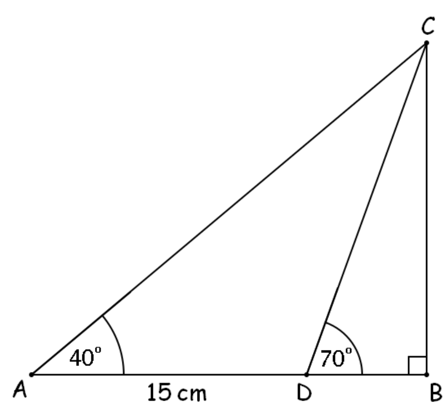
We now state just one such equality and cancel out a few things.

$$\begin{aligned} \frac{1}{2}ab\sin\gamma &= \frac{1}{2}ac\sin\beta && \text{divide by } \frac{1}{2}a \\ b\sin\gamma &= c\sin\beta && \text{divide by } bc \\ \frac{\sin\gamma}{c} &= \frac{\sin\beta}{b} \end{aligned} \qquad \text{and} \qquad \begin{aligned} \frac{1}{2}ac\sin\beta &= \frac{1}{2}bc\sin\alpha && \text{divide by } \frac{1}{2}c \\ a\sin\beta &= b\sin\alpha && \text{divide by } ab \\ \frac{\sin\beta}{b} &= \frac{\sin\alpha}{a} \end{aligned}$$

Thus we have proved that

$$\frac{\sin\alpha}{a} = \frac{\sin\beta}{b} = \frac{\sin\gamma}{c}$$

3. Compute the length of line segment BD based on the picture shown.



Solution: Angle $ADC = 110°$. Now we can compute every side and angle in triangle ADC using the law of sines.

$$\measuredangle ACD = 180° - (40° + 110°) = 30°$$

We will now compute the length of line segment CD.

$$\frac{CD}{\sin 40°} = \frac{15\text{ cm}}{\sin 30°} \quad \Longrightarrow \quad CD = \frac{15\text{ cm}\sin 40°}{\sin 30°} \approx 19.2836283\text{ cm}$$

We will now use right triangle trigonometry to compute the length of line segment BD.

$$\cos 70° = \frac{BD}{CD} \quad \Longrightarrow \quad BD = CD\cos 70° \approx 19.2836283\text{ cm}\cos 70° \approx 6.59539\text{ cm}$$

4. Prove that in any triangle, $P = \dfrac{a}{\sin\alpha}(\sin\alpha + \sin\beta + \sin\gamma)$.

Proof: The perimeter of any triangle is the sum of the lengths of its three sides. The law of sines will enable us to eliminate some sides from the formula.

$$\begin{aligned} P &= a+b+c = a\cdot\frac{\sin\alpha}{\sin\alpha} + b\cdot\frac{\sin\beta}{\sin\beta} + c\cdot\frac{\sin\gamma}{\sin\gamma} \\ &= \frac{a}{\sin\alpha}\cdot\sin\alpha + \frac{b}{\sin\beta}\cdot\sin\beta + \frac{c}{\sin\gamma}\cdot\sin\gamma \qquad \text{by the law of sines, } \frac{a}{\sin\alpha} = \frac{b}{\sin\beta} = \frac{c}{\sin\gamma} \\ &= \frac{a}{\sin\alpha}\cdot\sin\alpha + \frac{a}{\sin\alpha}\cdot\sin\beta + \frac{a}{\sin\alpha}\cdot\sin\gamma = \frac{a}{\sin\alpha}(\sin\alpha+\sin\beta+\sin\gamma) \end{aligned}$$

5. Solve the given triangles. Present a decimal approximation of all answers, accurate up to four or more decimal places.

a) $a = 3$, $\alpha = 42°$, $\beta = 100°$

Solution: Since two angles were given, we can easily compute the third.

$$\alpha+\beta+\gamma = 180° \quad\Longrightarrow\quad \gamma = 180° - (\alpha+\beta) = 180° - (42° + 100°) = 38°$$

Now that we know all angles and one side, the Law of Sines will enable us to find the length of all sides. To compute b, we state the Law of Sines for a, b, and α and β and solve for b. To make the algebra easier, we will chose a form of the theorem where the unknown, b is in the numerator.

$$\frac{b}{\sin\beta} = \frac{a}{\sin\alpha} \quad\Longrightarrow\quad b = \frac{a\sin\beta}{\sin\alpha} = \frac{3\sin 100°}{\sin 42°} \approx 4.415\,316\,28$$

We find c similarly: we state the Law of Sines for a, c, and α and γ and solve for c.

$$\frac{c}{\sin\gamma} = \frac{a}{\sin\alpha} \quad\Longrightarrow\quad c = \frac{a\sin\gamma}{\sin\alpha} = \frac{3\sin 38°}{\sin 42°} \approx 2.760\,275$$

To solve a triangle means finding all sides and angles that were not given. Thus the answer is: $\gamma = 38°$, $b \approx 4.415\,316\,28$ unit, and $c \approx 2.760\,275$ unit.

b) $a = 10$, $b = 12$, $\alpha = 32°$

Solution: The only thing we can compute from this data is $\sin\beta$. We will do exactly that: we state the Law of Sines between a, b, α and β and solve for $\sin\beta$.

$$\frac{\sin\alpha}{a} = \frac{\sin\beta}{b} \quad\Longrightarrow\quad \sin\beta = \frac{b\sin\alpha}{a} = \frac{12\sin 32°}{10} \approx 0.635\,903$$

Next we will solve for β. Recall that for all angles θ, $\sin\theta = \sin(180° - \theta)$ and so there are two possible solutions between $0°$ and $180°$. When we enter $\sin^{-1}(0.635\,903)$ into the calculator, we obtain only one of the possible solutions. The other one is the complement of the first one.

$$\sin\beta = 0.635\,903 \quad\Longrightarrow\quad \begin{cases} \beta_1 \approx 39.486\,999° \\ \beta_2 \approx 140.513\,001° \end{cases}$$

Sometimes both values work. From now on, we perform every step twice.

Now that we know β, we can compute the third angle, γ.

$$\begin{aligned}\gamma_1 &= 180^\circ - (\alpha+\beta_1) = 180^\circ - (32^\circ + 39.486\,999^\circ) = 108.513\,001^\circ \\ \gamma_2 &= 180^\circ - (\alpha+\beta_2) = 180^\circ - (32^\circ + 140.513\,001^\circ) = 7.486\,999^\circ\end{aligned}$$

Finally, we wil compute the length of side c. We will use the Law of Sines again.

$$\frac{c}{\sin\gamma} = \frac{a}{\sin\alpha} \quad \Longrightarrow \quad c = \frac{a\sin\gamma}{\sin\alpha}$$

The two possible values of c are then

$$\begin{aligned}c_1 &= \frac{a\sin\gamma_1}{\sin\alpha} = \frac{10\sin 108.513\,001^\circ}{\sin 32^\circ} \approx 17.894\,266 \\ c_2 &= \frac{a\sin\gamma_2}{\sin\alpha} = \frac{10\sin 7.486\,999^\circ}{\sin 32^\circ} \approx 2.458\,888\end{aligned}$$

There are two solutions:

$\beta_1 = 39.487^\circ$, $\gamma_1 = 108.513^\circ$, $c_1 = 17.894$ and $\beta_2 = 140.513^\circ$, $\gamma_2 = 7.487^\circ$, $c_2 = 2.459$

c) $a = 7 \quad b = 9 \quad \alpha = 114^\circ$

Solution: If we carefully look at the data given, we might see right away that there is something wrong. Because a is less than b, we also have that α is less than β. But $\alpha = 114^\circ$ and β even greater is impossible. Let us see how the Law of Sines works. Because a, b, and α are given, we can solve for $\sin\beta$.

$$\frac{\sin\alpha}{a} = \frac{\sin\beta}{b} \quad \Longrightarrow \quad \sin\beta = \frac{b\sin\alpha}{a} = \frac{9\sin 114^\circ}{7} \approx 1.174\,56$$

The equation $\sin\beta \approx 1.174\,56$ has no solution because the sine of any angle is less than or equal to 1. This result is telling us that there is no triangle with the data given above.

d) $a = 22$ cm, $b = 23$ cm, $\alpha = 67^\circ$

Solution: We can solve first for $\sin\beta$ by using the law of sines.

$$\frac{\sin\alpha}{a} = \frac{\sin\beta}{b} \quad \Longrightarrow \quad \sin\beta = \frac{b\sin\alpha}{a} = \frac{23\text{ cm}\sin 67^\circ}{22\text{ cm}} \approx 0.962\,346$$

The equation $\sin\beta = 0.962346$ has two solutions between 0° and 180°, and they are complements.

$$\sin\beta = 0.962\,346 \quad \Longrightarrow \quad \begin{cases}\beta_1 \approx 74.226955^\circ \\ \beta_2 \approx 105.773\,045^\circ\end{cases}$$

Let us now compute γ.

$$\begin{aligned}\gamma_1 &= 180^\circ - (\alpha+\beta_1) = 180^\circ - (67^\circ + 74.226955^\circ) = 38.773\,045^\circ \\ \gamma_2 &= 180^\circ - (\alpha+\beta_2) = 180^\circ - (67^\circ + 105.773\,045^\circ) = 7.226\,955^\circ\end{aligned}$$

In this case both values work. From now on, we perform every step twice. We can now compute c via the law of sines.

$$\frac{a}{\sin\alpha}=\frac{c}{\sin\gamma}\quad\Longrightarrow\quad c=\frac{a\sin\gamma}{\sin\alpha}$$

The two possible values of c are then

$$\begin{aligned} c_1 &= \frac{a\sin\gamma_1}{\sin\alpha}=\frac{22\text{ cm}\sin 38.773045^\circ}{\sin 67^\circ}\approx 14.96702\text{ cm}\\ c_2 &= \frac{a\sin\gamma_2}{\sin\alpha}=\frac{22\text{ cm}\sin 7.226\,955^\circ}{\sin 67^\circ}\approx 3.00661\text{ cm}\end{aligned}$$

Thus the two solutions are

$\beta_1\approx 74.227^\circ$, $\gamma_1\approx 38.773\,0^\circ$, $c_1\approx 14.9670$ cm and $\beta_2\approx 105.773^\circ$, $\gamma_2\approx 7.2270^\circ$, $c_2\approx 3.0066$ cm

e) $a=22$ cm, $b=23$ cm, $\beta=67^\circ$

Solution: We can solve first for $\sin\alpha$ by using the law of sines.

$$\frac{\sin\alpha}{a}=\frac{\sin\beta}{b}\quad\Longrightarrow\quad \sin\alpha=\frac{a\sin\beta}{b}=\frac{22\text{ cm}\sin 67^\circ}{23\text{ cm}}\approx 0.880\,482\,9$$

The equation $\sin\alpha=0.880482\,9$ has two solutions between 0° and 180°, and they are complements.

$$\sin\alpha=0.880\,482\,9\quad\Longrightarrow\quad\begin{cases}\alpha_1\approx 61.700\,67^\circ\\ \alpha_2\approx 118.299\,33^\circ\end{cases}$$

Let us now compute γ.

$$\begin{aligned}\gamma_1 &= 180^\circ-(\alpha_1+\beta)=180^\circ-(67^\circ+61.700\,67^\circ)=51.299\,33^\circ\\ \gamma_2 &= 180^\circ-(\alpha_2+\beta)=180^\circ-(67^\circ+118.299\,33^\circ)=-5.299\,33^\circ\end{aligned}$$

Since the sum of β and α_2 is greater than 180°, the numbers here are telling us that there is only one triangle with the data given. So, $\gamma\approx 51.299\,33^\circ$. We can now compute c via the law of sines.

$$\frac{c}{\sin\gamma}=\frac{b}{\sin\beta}\quad\Longrightarrow\quad c=\frac{b\sin\gamma}{\sin\beta}=\frac{23\text{ cm}\sin 51.299\,33^\circ}{\sin 67^\circ}\approx 19.499\,877\text{ cm}$$

Thus the solution is $\alpha\approx 61.700\,67^\circ$, $\gamma\approx 51.299\,33^\circ$, and $c\approx 19.499\,877$ cm.

f) $a=6$, $b=12$, $\alpha=30^\circ$

Solution: We can solve first for $\sin\beta$ by using the law of sines.

$$\frac{\sin\alpha}{a}=\frac{\sin\beta}{b}\quad\Longrightarrow\quad \sin\beta=\frac{b\sin\alpha}{a}=\frac{12\sin 30^\circ}{6}=1$$

The equation $\sin\beta=1$ has only one solution between 0° and 180°, it is $\beta=90^\circ$. Let us now compute γ.

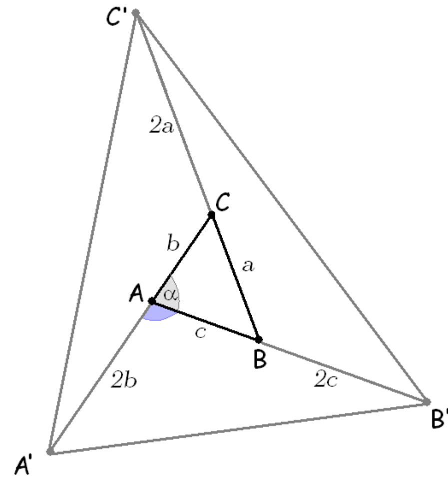
$$\gamma=180^\circ-(\alpha+\beta)=180^\circ-(30^\circ+90^\circ)=60^\circ$$

In this case the triangle has a right angle and so we can compute c by applying the Pythagorean theorem. Since $\beta=90^\circ$, the hypotenuse is b.

$$c=\sqrt{b^2-a^2}=\sqrt{12^2-6^2}=\sqrt{108}=6\sqrt{3}$$

Thus the solution is $\beta=90^\circ$, $\gamma=60^\circ$, and $c=6\sqrt{3}\approx 10.392\,305$

6. Let ABC be any triangle. We extend the line segments AB, BC, and CA as shown on picture. Suppose that $\overline{AA'} = 2\overline{AC}$, and $\overline{BB'} = 2\overline{AB}$, and $\overline{CC'} = 2\overline{BC}$. What is the ratio of the area of triangle $A'B'C'$ to the area of triangle ABC?



Solution: The angle (marked blue) at point A is $180° - \alpha$. Similarly, angle CBB' is $180° - \beta$, and the angle $C'CA$ is $180° - \gamma$. Let A denote the area of the original triangle ABC.

Also recall that $\sin(180° - \alpha) = \sin\alpha$ for all angles α.

$$A = \frac{1}{2}ab\sin\gamma = \frac{1}{2}ac\sin\beta = \frac{1}{2}bc\sin\alpha$$

We will compute the area of triangle $A'B'C'$ as the sum of the areas of four triangles: $\triangle ABC$, $\triangle A'C'C$, $\triangle C'BB'$, and $\triangle AA'B'$.

$$A_{CC'A'} = \frac{1}{2}\cdot 2a\cdot 3b\cdot\sin(180° - \gamma) = \frac{1}{2}6ab\sin\gamma = 6\cdot\left(\frac{1}{2}ab\sin\gamma\right) = 6A$$

$$A_{C'B'B} = \frac{1}{2}\cdot 2c\cdot 3a\cdot\sin(180° - \beta) = \frac{1}{2}6ac\sin\beta = 6\cdot\left(\frac{1}{2}ac\sin\beta\right) = 6A$$

$$A_{AB'A'} = \frac{1}{2}\cdot 2b\cdot 3c\cdot\sin(180° - \alpha) = \frac{1}{2}6bc\sin\alpha = 6\cdot\left(\frac{1}{2}bc\sin\alpha\right) = 6A$$

So the area of triangle $A'B'C'$ is $A = A + 6A + 6A + 6A = 19A$. So the ratio of the area of triangle $A'B'C'$ to the area of triangle ABC is $\boxed{\text{19 to 1}}$.

