Sample Problems

1. Express each of the following products as a sum or a difference.

   a) $\sin 5x \cos 12x$ b) $\cos 24A \cos 8A$ c) $\sin 8\alpha \cos 2\alpha$ d) $\sin 7\theta \sin 4\theta$

2. Re-write each of the following sums as a product.

   a) $\sin 13a - \sin 3a$ b) $\cos 8\beta + \cos 20\beta$

3. Simplify the given expression. $\dfrac{\cos 3A - \cos 11A}{\sin 9A + \sin 5A}$

4. Prove that $\sin 20^\circ \sin 40^\circ \sin 60^\circ \sin 80^\circ = \dfrac{3}{16}$

Practice Problems

1. Express each of the following products as a sum or a difference.

   a) $\sin 8A \cos 6A$ c) $\cos 8A \cos 6A$ e) $\sin 3\alpha \cos 19\alpha$ g) $\sin 6m \sin 3m$

   b) $\sin 6A \cos 8A$ d) $\sin 8A \sin 6A$ f) $\cos 42\theta \cos 6\theta$ h) $\sin \alpha \cos \beta$

2. Re-write each of the following sums as a product.

   a) $\sin 10M + \sin 4M$ b) $\sin 10M - \sin 4M$ c) $\cos 5x + \cos 21x$ d) $\cos 21x - \cos 5x$

3. Simplify the given expression. $\dfrac{\cos 4x - \cos 6x}{\sin 6x - \sin 4x}$

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Sample Problems - Answers

1. a) $\frac{1}{2}\sin 17x - \frac{1}{2}\sin 7x$ b) $\frac{1}{2}\cos 32A + \frac{1}{2}\cos 16A$ c) $\frac{1}{2}\sin 10\alpha + \frac{1}{2}\sin 6\alpha$
   d) $\frac{1}{2}\cos 3\theta - \frac{1}{2}\cos 11\theta$
2. a) $2\cos 8a \sin 5a$ b) $2\cos 14\beta \cos 6\beta$ 3. $2\sin 2A$

Practice Problems - Answers

1. a) $\frac{1}{2}\sin 14A + \frac{1}{2}\sin 2A$ b) $\frac{1}{2}\sin 14A - \frac{1}{2}\sin 2A$ c) $\frac{1}{2}\cos 14A + \frac{1}{2}\cos 2A$
   d) $\frac{1}{2}\cos 2A - \frac{1}{2}\cos 14A$ e) $\frac{1}{2}\sin 22\alpha - \frac{1}{2}\sin 16\alpha$ f) $\frac{1}{2}\cos 36\theta + \frac{1}{2}\cos 48\theta$
   g) $\frac{1}{2}\cos 3m - \frac{1}{2}\cos 9m$ h) $\frac{1}{2}\sin(\alpha+\beta) + \frac{1}{2}\sin(\alpha-\beta)$
2. a) $2\sin 7M \cos 3M$ b) $2\sin 3M \cos 7M$ c) $2\cos 13x \cos 8x$ d) $-2\sin 13x \sin 8x$
3. $\tan 5x$

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Sample Problems - Solutions

1. Express each of the following products as a sum or a difference.

a) $\sin 5x \cos 12x$

We will write the sum and difference formula for sine using $5x$ and $12x$.

$$\begin{aligned} \sin 17x &= \sin(12x + 5x) = \sin 12x \cos 5x + \cos 12x \sin 5x \\ \sin 7x &= \sin(12x - 5x) = \sin 12x \cos 5x - \cos 12x \sin 5x \end{aligned}$$

We will subtract the second equation from the first one - or rather, add the opposite.

$$\begin{aligned} &\sin 17x = \sin 12x \cos 5x + \cos 12x \sin 5x \\ +\quad &-\sin 7x = -\sin 12x \cos 5x + \cos 12x \sin 5x \\ \hline &\sin 17x - \sin 7x = 2\cos 12x \sin 5x \\ &\frac{1}{2}(\sin 17x - \sin 7x) = \cos 12x \sin 5x \end{aligned}$$

and so the answer is $\cos 12x \sin 5x = \boxed{\frac{1}{2}\sin 17x - \frac{1}{2}\sin 7x}$

b) $\cos 24A \cos 8A$

We will write the sum and difference formula for cosine using $8A$ and $24A$.

$$\begin{aligned} \cos 32A &= \cos(24A + 8A) = \cos 24A \cos 8A - \sin 24A \sin 8A \\ \cos 16A &= \cos(24A - 8A) = \cos 24A \cos 8A + \sin 24A \sin 8A \end{aligned}$$

We will add the two equations

$$\begin{aligned} &\cos 32A = \cos 24A \cos 8A - \sin 24A \sin 8A \\ +\quad &\cos 16A = \cos 24A \cos 8A + \sin 24A \sin 8A \\ \hline &\cos 32A + \cos 16A = 2\cos 24A \cos 8A \\ &\frac{1}{2}(\cos 32A + \cos 16A) = \cos 24A \cos 8A \end{aligned}$$

and so the answer is $\cos 24A \cos 8A = \boxed{\frac{1}{2}\cos 32A + \frac{1}{2}\cos 16A}$

c) $\sin 8\alpha \cos 2\alpha$

We will write the sum and difference formula for sine using 8α and 2α.

$$\begin{aligned} \sin 10\alpha &= \sin(8\alpha + 2\alpha) = \sin 8\alpha \cos 2\alpha + \cos 8\alpha \sin 2\alpha \\ \sin 6\alpha &= \sin(8\alpha - 2\alpha) = \sin 8\alpha \cos 2\alpha - \cos 8\alpha \sin 2\alpha \end{aligned}$$

We will add the two equations.

$$\begin{aligned} &\sin 10\alpha = \sin 8\alpha \cos 2\alpha + \cos 8\alpha \sin 2\alpha \\ +\quad &\sin 6\alpha = \sin 8\alpha \cos 2\alpha - \cos 8\alpha \sin 2\alpha \\ \hline &\sin 10\alpha + \sin 6\alpha = 2\sin 8\alpha \cos 2\alpha \\ &\frac{1}{2}(\sin 10\alpha + \sin 6\alpha) = \sin 8\alpha \cos 2\alpha \end{aligned}$$

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and so the answer is $\sin 8\alpha \cos 2\alpha = \boxed{\frac{1}{2}(\sin 10\alpha + \sin 6\alpha)}$

d) $\sin 7\theta \sin 4\theta$

We will write the sum and difference formula for cosine using 7θ and 4θ.

$$\begin{aligned} \cos 11\theta &= \cos(7\theta + 4\theta) = \cos 7\theta \cos 4\theta - \sin 7\theta \sin 4\theta \\ \cos 3\theta &= \cos(7\theta - 4\theta) = \cos 7\theta \cos 4\theta + \sin 7\theta \sin 4\theta \end{aligned}$$

We will subtract the first equation from the second - or rather, add the opposite:

$$\begin{aligned} -\cos 11\theta &= -(\cos 7\theta \cos 4\theta + \sin 7\theta \sin 4\theta) \\ + \quad \cos 3\theta &= \cos(7\theta - 4\theta) = \cos 7\theta \cos 4\theta + \sin 7\theta \sin 4\theta \\ \hline \cos 3\theta - \cos 11\theta &= 2\sin 7\theta \sin 4\theta \\ \sin 7\theta \sin 4\theta &= \frac{1}{2}(\cos 3\theta - \cos 11\theta) \end{aligned}$$

and so the answer is $\sin 7\theta \sin 4\theta = \boxed{\frac{1}{2}(\cos 3\theta - \cos 11\theta)}$

2. Re-write each of the following sums as a product.

a) $\sin 13a - \sin 3a$

Solution: We will need to come up with two angles that add up to $13a$ and their difference is $3a$. This is quite easy, especially if we just remember that one of those angles must be the average of $13a$ and $3a$. This average is $\frac{13a + 3a}{2} = 8a$. Now the other angle is $13a - 8a = 5a$. Indeed, the sum of $5a$ and $8a$ is $13a$ and their difference is $3a$. There are no more decisions to make. Looking at the problem posed, we know that we will express the sum and difference formulas for sine; and that we will subtract $\sin 3a$ from $\sin 13a$.

$$\begin{aligned} \sin 13a &= \sin(8a + 5a) = \sin 8a \cos 5a + \cos 8a \sin 5a \\ \sin 3a &= \sin(8a - 5a) = \sin 8a \cos 5a - \cos 8a \sin 5a \end{aligned}$$

We will subtract $\sin 3a$ from $\sin 13a$ - or rather, add the opposite:

$$\begin{aligned} \sin 13a = \sin(8a + 5a) &= \sin 8a \cos 5a + \cos 8a \sin 5a \\ + \quad -\sin 3a = -\sin(8a - 5a) &= -\sin 8a \cos 5a + \cos 8a \sin 5a \\ \hline \sin 13a - \sin 3a &= 2\cos 8a \sin 5a \end{aligned}$$

and so the answer is $\sin 13a - \sin 3a = \boxed{2\cos 8a \sin 5a}$

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b) $\cos 8\beta + \cos 20\beta$

Solution: We will need to come up with two angles that add up to 20β and their difference is 8β. This is quite easy, especially if we just remember that one of those angles must be the average these two, $\dfrac{8\beta + 20\beta}{2} = 14\beta$. Now the other angle is $20\beta - 14\beta = 6\beta$. Looking at the problem given, we know that we will express the sum and difference formulas for cosine; and that we will add $\cos 8\beta$ and $\cos 20\beta$.

$$\begin{aligned} \cos 20\beta &= \cos(14\beta + 6\beta) = \cos 14\beta \cos 6\beta - \sin 14\beta \sin 6\beta \\ \cos 8\beta &= \cos(14\beta - 6\beta) = \cos 14\beta \cos 6\beta + \sin 14\beta \sin 6\beta \end{aligned}$$

We will add the two equations

$$\begin{array}{rl} & \cos 20\beta = \cos 14\beta \cos 6\beta - \sin 14\beta \sin 6\beta \\ + & \cos 8\beta = \cos 14\beta \cos 6\beta + \sin 14\beta \sin 6\beta \\ \hline & \cos 8\beta + \cos 20\beta = 2\cos 14\beta \cos 6\beta \end{array}$$

and so the answer is $\cos 8\beta + \cos 20\beta = \boxed{2\cos 14\beta \cos 6\beta}$

3. Simplify the given expression. $\dfrac{\cos 3A - \cos 11A}{\sin 9A + \sin 5A}$

Solution: Let us use the sum-product identities to turn the numerator and denominator into products. Then, if there is a common factor, we can cancel it out.

If we need to find two quantities with sum $11A$ and difference $3A$, let's take the average of the two first. We obtain $7A$ and then easily $4A$. Now we will state the sum and difference formula of cosine with angles $7A$ and $4A$.

$$\cos 11A = \cos(7A + 4A) = \cos 7A \cos 4A - \sin 7A \sin 4A$$

$$\cos 3A = \cos(7A - 4A) = \cos 7A \cos 4A + \sin 7A \sin 4A$$

Since we want to express $\cos 3A - \cos 11A$, we subtract the equations accordingly.

$$\begin{array}{rl} & -\cos 11A = -\cos 7A \cos 4A + \sin 7A \sin 4A \\ + & \cos 3A = \cos 7A \cos 4A + \sin 7A \sin 4A \\ \hline & \cos 3A - \cos 11A = 2\sin 7A \sin 4A \end{array}$$

so the numerator can be re-written as $2\sin 7A \sin 4A$.

We repeat the process with the denominator. First we need to find two quantities with sum $9A$ and difference $5A$. We find $7A$ and then easily $2A$. Now we will state the sum and difference formula of sine with angles $7A$ and $2A$.

$$\sin 9A = \sin(7A + 2A) = \sin 7A \cos 2A + \cos 7A \sin 2A$$

$$\sin 5A = \sin(7A - 2A) = \sin 7A \cos 2A - \cos 7A \sin 2A$$

We add the two equations and obtain $\sin 9A + \sin 5A = 2\sin 7A \cos 2A$. So the denominator can be re-written as $2\sin 7A \cos 2A$. So our fraction is now

$$\frac{\cos 3A - \cos 11A}{\sin 9A + \sin 5A} = \frac{2\sin 7A \sin 4A}{2\sin 7A \cos 2A} = \frac{\sin 4A}{\cos 2A}$$

This expression can be further simplified, via the double-angle formula for sine: $\sin 4A = 2\sin 2A \cos 2A$.

$$\frac{\sin 4A}{\cos 2A} = \frac{2\sin 2A \cos 2A}{\cos 2A} = 2\sin 2A$$

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So the answer is $\boxed{2\sin 2A}$.

4. Prove that $\sin 20^\circ \sin 40^\circ \sin 60^\circ \sin 80^\circ = \dfrac{3}{16}$

Solution: We will apply the product-sum identities several times. First for $\sin 20^\circ \sin 40^\circ$ and for $\sin 60^\circ \sin 80^\circ$.

$$\begin{aligned} \cos 20^\circ = \cos\left(40^\circ - 20^\circ\right) &= \cos 40^\circ \cos 20^\circ + \sin 40^\circ \sin 20^\circ \\ \cos 60^\circ = \cos\left(40^\circ + 20^\circ\right) &= \cos 40^\circ \cos 20^\circ - \sin 40^\circ \sin 20^\circ \end{aligned}$$

If we subtract the second equation from the first one, we get that $\cos 20^\circ - \cos 60^\circ = 2\sin 40^\circ \sin 20^\circ$

and so $\sin 40^\circ \sin 20^\circ = \dfrac{1}{2}\left(\cos 20^\circ - \cos 60^\circ\right)$

Similarly,

$$\begin{aligned} \cos 20^\circ = \cos\left(80^\circ - 60^\circ\right) &= \cos 80^\circ \cos 60^\circ + \sin 80^\circ \sin 60^\circ \\ \cos 140^\circ = \cos\left(80^\circ + 60^\circ\right) &= \cos 80^\circ \cos 60^\circ - \sin 80^\circ \sin 60^\circ \end{aligned}$$

If we subtract the second equation from the first one, we get that $\cos 20^\circ - \cos 140^\circ = 2\sin 60^\circ \sin 80^\circ$

and so $\sin 60^\circ \sin 80^\circ = \dfrac{1}{2}\left(\cos 20^\circ - \cos 140^\circ\right)$

Let us re-write $\cos 140^\circ$ as $-\cos 40^\circ$ and then we have that $\sin 60^\circ \sin 80^\circ = \dfrac{1}{2}\left(\cos 20^\circ + \cos 40^\circ\right)$

$$\begin{aligned} \sin 20^\circ \sin 40^\circ \sin 60^\circ \sin 80^\circ &= \frac{1}{2}\left(\cos 20^\circ - \cos 60^\circ\right)\frac{1}{2}\left(\cos 20^\circ + \cos 40^\circ\right) \\ &= \frac{1}{4}\left(\cos 20^\circ - \cos 60^\circ\right)\left(\cos 20^\circ + \cos 40^\circ\right) \\ &= \frac{1}{4}\left(\cos^2 20^\circ + \cos 20^\circ \cos 40^\circ - \cos 20^\circ \cos 60^\circ - \cos 40^\circ \cos 60^\circ\right) \end{aligned}$$

Now we will apply the product-sum formulas again, except for the first term, $\cos^2 20^\circ$ which we will handle by the double angle formula.

$\cos 40^\circ = 2\cos^2 20^\circ - 1$ and so $\cos^2 20^\circ = \dfrac{1}{2}\left(\cos 40^\circ + 1\right)$

Now we re-write $\cos 20^\circ \cos 40^\circ$

$$\begin{aligned} \cos 20^\circ = \cos\left(40^\circ - 20^\circ\right) &= \cos 40^\circ \cos 20^\circ + \sin 40^\circ \sin 20^\circ \\ \cos 60^\circ = \cos\left(40^\circ + 20^\circ\right) &= \cos 40^\circ \cos 20^\circ - \sin 40^\circ \sin 20^\circ \end{aligned}$$

If we add the two equations, we get that $\cos 20^\circ + \cos 60^\circ = 2\cos 40^\circ \cos 20^\circ$

and so $\cos 20^\circ \cos 40^\circ = \dfrac{1}{2}\left(\cos 20^\circ + \cos 60^\circ\right)$

Similarly, we can re-write $\cos 20^\circ \cos 60^\circ$ as $\dfrac{1}{2}\left(\cos 40^\circ + \cos 80^\circ\right)$ and $\cos 40^\circ \cos 60^\circ$ as $\dfrac{1}{2}\left(\cos 20^\circ + \cos 100^\circ\right)$

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So we have that $\sin 20^\circ \sin 40^\circ \sin 60^\circ \sin 80^\circ =$

$$\begin{aligned} &= \frac{1}{4}\left(\cos^2 20^\circ + \cos 20^\circ \cos 40^\circ - \cos 20^\circ \cos 60^\circ - \cos 40^\circ \cos 60^\circ\right) \\ &= \frac{1}{4}\left(\frac{1}{2}(1+\cos 40^\circ) + \frac{1}{2}(\cos 20^\circ + \cos 60^\circ) - \frac{1}{2}(\cos 40^\circ + \cos 80^\circ) - \frac{1}{2}(\cos 20^\circ + \cos 100^\circ)\right) \\ &= \frac{1}{8}(1 + \cos 40^\circ + \cos 20^\circ + \cos 60^\circ - \cos 40^\circ - \cos 80^\circ - \cos 20^\circ - \cos 100^\circ) \end{aligned}$$

We can cancel out $\cos 20^\circ$ and $\cos 40^\circ$ and have that

$$\sin 20^\circ \sin 40^\circ \sin 60^\circ \sin 80^\circ = \frac{1}{8}(1 + \cos 60^\circ - \cos 80^\circ - \cos 100^\circ)$$

We know that $\cos 60^\circ = \frac{1}{2}$ and that $\cos 100^\circ = -\cos 80^\circ$ and so

$$\sin 20^\circ \sin 40^\circ \sin 60^\circ \sin 80^\circ = \frac{1}{8}\left(1 + \frac{1}{2} - \cos 80^\circ - (-\cos 80^\circ)\right) = \frac{1}{8} \cdot \frac{3}{2} = \frac{3}{16}$$

This completes our proof.

5. Prove that $\cos 20^\circ + \cos 60^\circ + \cos 100^\circ + \cos 140^\circ = \frac{1}{2}$

Solution: We will multiply this expression by $\frac{2\sin 20^\circ}{2\sin 20^\circ}$. Then we will distribute the numerator and then apply the product-sum identities to re-write the products.

$\cos 20^\circ + \cos 60^\circ + \cos 100^\circ + \cos 140^\circ =$

$$\begin{aligned} &= \frac{2\sin 20^\circ}{2\sin 20^\circ}(\cos 20^\circ + \cos 60^\circ + \cos 100^\circ + \cos 140^\circ) \\ &= \frac{1}{2\sin 20^\circ}(2\sin 20^\circ \cos 20^\circ + 2\sin 20^\circ \cos 60^\circ + 2\sin 20^\circ \cos 100^\circ + 2\sin 20^\circ \cos 140^\circ) \end{aligned}$$

The first term can be simplified by the double angle formula for sine: $2\sin 20^\circ \cos 20^\circ = \sin 40^\circ$

Now we re-write $2\sin 20^\circ \cos 60^\circ$

$$\begin{aligned} \sin 80^\circ = \sin(60^\circ + 20^\circ) &= \sin 60^\circ \cos 20^\circ + \cos 60^\circ \sin 20^\circ \\ \sin 40^\circ = \sin(60^\circ - 20^\circ) &= \sin 60^\circ \cos 20^\circ - \cos 60^\circ \sin 20^\circ \end{aligned}$$

Let us subtract the second equation from the first and get $\sin 80^\circ - \sin 40^\circ = 2\cos 60^\circ \sin 20^\circ$

Similarly, we can re-write $2\cos 100^\circ \sin 20^\circ$ as $\sin 120^\circ - \sin 80^\circ$ and $2\cos 140^\circ \sin 20^\circ$ as $\sin 160^\circ - \sin 120^\circ$

So then we will have that

$$\begin{aligned} &\frac{1}{2\sin 20^\circ}(2\sin 20^\circ \cos 20^\circ + 2\sin 20^\circ \cos 60^\circ + 2\sin 20^\circ \cos 100^\circ + 2\sin 20^\circ \cos 140^\circ) \\ &= \frac{1}{2\sin 20^\circ}(\sin 40^\circ + \sin 80^\circ - \sin 40^\circ + \sin 120^\circ - \sin 80^\circ + \sin 160^\circ - \sin 120^\circ) \end{aligned}$$

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There are cancellations: $\sin 40°,\ \sin 80°,$ and $\sin 120°$ will be canceled out and we are left with $\dfrac{1}{2\sin 20°}(\sin 160°)$

Since they are supplements, $20°$ and $160°$ have the same sine, and so $\dfrac{1}{2\sin 20°}(\sin 160°) = \dfrac{\sin 20°}{2\sin 20°} = \dfrac{1}{2}$.

This completes our proof.

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