

Sample Problems

1. Express each of the following products as a sum or a difference.

a) $\sin 5x \cos 12x$

b) $\cos 24A \cos 8A$

c) $\sin 8\alpha \cos 2\alpha$

d) $\sin 7\theta \sin 4\theta$

2. Re-write each of the following sums as a product.

a) $\sin 13a - \sin 3a$

b) $\cos 8 + \cos 20\beta$

3. Simplify the given expression. $\frac{\cos 3A - \cos 11A}{\sin 9A + \sin 5A}$

4. Prove that $\sin 20^\circ \sin 40^\circ \sin 60^\circ \sin 80^\circ = \frac{3}{16}$

Practice Problems

1. Express each of the following products as a sum or a difference.

a) $\sin 8A \cos 6A$

c) $\cos 8A \cos 6A$

e) $\sin 3\alpha \cos 19\alpha$

g) $\sin 6m \sin 3m$

b) $\sin 6A \cos 8A$

d) $\sin 8A \sin 6A$

f) $\cos 42\theta \cos 6\theta$

h) $\sin \alpha \cos \beta$

2. Re-write each of the following sums as a product.

a) $\sin 10M + \sin 4M$

b) $\sin 10M - \sin 4M$

c) $\cos 5x + \cos 21x$

d) $\cos 21x - \cos 5x$

3. Simplify the given expression. $\frac{\cos 4x - \cos 6x}{\sin 6x - \sin 4x}$

Sample Problems - Answers

1. a) $\frac{1}{2} \sin 17x - \frac{1}{2} \sin 7x$ b) $\frac{1}{2} \cos 32A + \frac{1}{2} \cos 16A$ c) $\frac{1}{2} \sin 10\alpha + \frac{1}{2} \sin 6\alpha$
 d) $\frac{1}{2} \cos 3\theta - \frac{1}{2} \cos 11\theta$
2. a) $2 \cos 8a \sin 5a$ b) $2 \cos 14\beta \cos 6\beta$ 3. $2 \sin 2A$

Practice Problems - Answers

1. a) $\frac{1}{2} \sin 14A + \frac{1}{2} \sin 2A$ b) $\frac{1}{2} \sin 14A - \frac{1}{2} \sin 2A$ c) $\frac{1}{2} \cos 14A + \frac{1}{2} \cos 2A$
 d) $\frac{1}{2} \cos 2A - \frac{1}{2} \cos 14A$ e) $\frac{1}{2} \sin 22\alpha - \frac{1}{2} \sin 16\alpha$ f) $\frac{1}{2} \cos 36\theta + \frac{1}{2} \cos 48\theta$
 g) $\frac{1}{2} \cos 3m - \frac{1}{2} \cos 9m$ h) $\frac{1}{2} \sin (\alpha + \beta) + \frac{1}{2} \sin (\alpha - \beta)$
2. a) $2 \sin 7M \cos 3M$ b) $2 \sin 3M \cos 7M$ c) $2 \cos 13x \cos 8x$ d) $-2 \sin 13x \sin 8x$
3. $\tan 5x$

Sample Problems - Solutions

1. Express each of the following products as a sum or a difference.

a) $\sin 5x \cos 12x$

We will write the sum and difference formula for sine using $5x$ and $12x$.

$$\sin 17x = \sin (12x + 5x) = \sin 12x \cos 5x + \cos 12x \sin 5x$$

$$\sin 7x = \sin (12x - 5x) = \sin 12x \cos 5x - \cos 12x \sin 5x$$

We will subtract the second equation from the first one - or rather, add the opposite.

$$\sin 17x = \sin 12x \cos 5x + \cos 12x \sin 5x$$

$$+ \quad - \sin 7x = -\sin 12x \cos 5x + \cos 12x \sin 5x$$

$$\sin 17x - \sin 7x = 2 \cos 12x \sin 5x$$

$$\frac{1}{2} (\sin 17x - \sin 7x) = \cos 12x \sin 5x$$

and so the answer is $\cos 12x \sin 5x = \boxed{\frac{1}{2} \sin 17x - \frac{1}{2} \sin 7x}$

b) $\cos 24A \cos 8A$

We will write the sum and difference formula for cosine using $8A$ and $24A$.

$$\cos 32A = \cos (24A + 8A) = \cos 24A \cos 8A - \sin 24A \sin 8A$$

$$\cos 16A = \cos (24A - 8A) = \cos 24A \cos 8A + \sin 24A \sin 8A$$

We will add the two equations

$$\cos 32A = \cos 24A \cos 8A - \sin 24A \sin 8A$$

$$+ \quad \cos 16A = \cos 24A \cos 8A + \sin 24A \sin 8A$$

$$\cos 32A + \cos 16A = 2 \cos 24A \cos 8A$$

$$\frac{1}{2} (\cos 32A + \cos 16A) = \cos 24A \cos 8A$$

and so the answer is $\cos 24A \cos 8A = \boxed{\frac{1}{2} \cos 32A + \frac{1}{2} \cos 16A}$

c) $\sin 8\alpha \cos 2\alpha$

We will write the sum and difference formula for sine using $8M$ and $2M$.

$$\sin 10\alpha = \sin (8\alpha + 2\alpha) = \sin 8\alpha \cos 2\alpha + \cos 8\alpha \sin 2\alpha$$

$$\sin 6\alpha = \sin (8\alpha - 2\alpha) = \sin 8\alpha \cos 2\alpha - \cos 8\alpha \sin 2\alpha$$

We will add the two equations.

$$\sin 10\alpha = \sin 8\alpha \cos 2\alpha + \cos 8\alpha \sin 2\alpha$$

$$+ \quad \sin 6\alpha = \sin 8\alpha \cos 2\alpha - \cos 8\alpha \sin 2\alpha$$

$$\sin 10\alpha + \sin 6\alpha = 2 \sin 8\alpha \cos 2\alpha$$

$$\frac{1}{2} (\sin 10\alpha + \sin 6\alpha) = \sin 8\alpha \cos 2\alpha$$

and so the answer is $\sin 8\alpha \cos 2\alpha = \boxed{\frac{1}{2}(\sin 10\alpha + \sin 6\alpha)}$

d) $\sin 7\theta \sin 4\theta$

We will write the sum and difference formula for cosine using $8A$ and $24A$.

$$\begin{aligned}\cos 11\theta &= \cos(7\theta + 4\theta) = \cos 7\theta \cos 4\theta - \sin 7\theta \sin 4\theta \\ \cos 3\theta &= \cos(7\theta - 4\theta) = \cos 7\theta \cos 4\theta + \sin 7\theta \sin 4\theta\end{aligned}$$

We will subtract the first equation from the second - or rather, add the opposite:

$$\begin{aligned}-\cos 11\theta &= -(\cos 7\theta \cos 4\theta + \sin 7\theta \sin 4\theta) \\ + \cos 3\theta &= \cos(7\theta - 4\theta) = \cos 7\theta \cos 4\theta + \sin 7\theta \sin 4\theta \\ \hline \cos 3\theta - \cos 11\theta &= 2 \sin 7\theta \sin 4\theta \\ \sin 7\theta \sin 4\theta &= \frac{1}{2}(\cos 3\theta - \cos 11\theta)\end{aligned}$$

and so the answer is $\sin 7\theta \sin 4\theta = \boxed{\frac{1}{2}(\cos 3\theta - \cos 11\theta)}$

2. Re-write each of the following sums as a product.

a) $\sin 13a - \sin 3a$

Solution: We will need to come up with two angles that add up to $13a$ and their difference is $3a$. This is quite easy, especially if we just remember that one of those angles must be the average of $13a$ and $3a$. This average is $\frac{13a + 3a}{2} = 8a$. Now the other angle is $13a - 8a = 5a$. Indeed, the sum of $5a$ and $8a$ is $13a$ and their difference is $3a$. There are no more decisions to make. Looking at the problem posed, we know that we will express the sum and difference formulas for sine; and that we will subtract $\sin 3a$ from $\sin 13a$.

$$\begin{aligned}\sin 13a &= \sin(8a + 5a) = \sin 8a \cos 5a + \cos 8a \sin 5a \\ \sin 3a &= \sin(8a - 5a) = \sin 8a \cos 5a - \cos 8a \sin 5a\end{aligned}$$

We will subtract $\sin 3a$ from $\sin 13a$ - or rather, add the opposite:

$$\begin{aligned}\sin 13a &= \sin(8a + 5a) = \sin 8a \cos 5a + \cos 8a \sin 5a \\ + -\sin 3a &= -\sin(8a - 5a) = -\sin 8a \cos 5a + \cos 8a \sin 5a \\ \hline \sin 13a - \sin 3a &= 2 \cos 8a \sin 5a\end{aligned}$$

and so the answer is $\sin 13a - \sin 3a = \boxed{2 \cos 8a \sin 5a}$

b) $\cos 8\beta + \cos 20\beta$

Solution: We will need to come up with two angles that add up to 20β and their difference is 8β . This is quite easy, especially if we just remember that one of those angles must be the average these two, $\frac{8\beta + 20\beta}{2} = 14\beta$. Now the other angle is $20\beta - 14\beta = 6\beta$. Looking at the problem given, we know that we will express the sum and difference formulas for cosine; and that we will add $\cos 8\beta$ and $\cos 20\beta$.

$$\begin{aligned}\cos 20\beta &= \sin(14\beta + 6\beta) = \cos 14\beta \cos 6\beta - \sin 14\beta \sin 6\beta \\ \cos 8\beta &= \sin(14\beta - 6\beta) = \cos 14\beta \cos 6\beta + \sin 14\beta \sin 6\beta\end{aligned}$$

We will add the two equations

$$\begin{aligned}\cos 20\beta &= \cos 14\beta \cos 6\beta - \sin 14\beta \sin 6\beta \\ + \cos 8\beta &= \cos 14\beta \cos 6\beta + \sin 14\beta \sin 6\beta \\ \hline \cos 8\beta + \cos 20\beta &= 2 \cos 14\beta \cos 6\beta\end{aligned}$$

and so the answer is $\cos 8\beta + \cos 20\beta = \boxed{2 \cos 14\beta \cos 6\beta}$

3. Simplify the given expression. $\frac{\cos 3A - \cos 11A}{\sin 9A + \sin 5A}$

Solution: Let us use the sum-product identities to turn the numerator and denominator into products. Then, if there is a common factor, we can cancel it out.

If we need to find two quantities with sum $11A$ and difference $3A$, let's take the average of the two first. We obtain $7A$ and then easily $4A$. Now we will state the sum and difference formula of cosine with angles $7A$ and $11A$.

$$\begin{aligned}\cos 11A &= \cos(7A + 4A) = \cos 7A \cos 4A - \sin 7A \sin 4A \\ \cos 3A &= \cos(7A - 4A) = \cos 7A \cos 4A + \sin 7A \sin 4A\end{aligned}$$

Since we want to express $\cos 3A - \cos 11A$, we subtract the equations accordingly.

$$\begin{aligned}-\cos 11A &= -\cos 7A \cos 4A + \sin 7A \sin 4A \\ + \cos 3A &= \cos 7A \cos 4A + \sin 7A \sin 4A \\ \hline \cos 3A - \cos 11A &= 2 \sin 7A \sin 4A\end{aligned}$$

so the numerator can be re-written as $2 \sin 7A \sin 4A$.

We repeat the process with the denominator. First we need to find two quantities with sum $9A$ and difference $5A$. We find $7A$ and then easily $2A$. Now we will state the sum and difference formula of sine with angles $7A$ and $2A$.

$$\begin{aligned}\sin 9A &= \sin(7A + 2A) = \sin 7A \cos 2A + \cos 7A \sin 2A \\ \sin 5A &= \sin(7A - 2A) = \sin 7A \cos 2A - \cos 7A \sin 2A\end{aligned}$$

We add the two equations and obtain $\sin 9A + \sin 5A = 2 \sin 7A \cos 2A$. So the denominator can be re-written as $2 \sin 7A \cos 2A$. So our fraction is now

$$\frac{\cos 3A - \cos 11A}{\sin 9A + \sin 5A} = \frac{2 \sin 7A \sin 4A}{2 \sin 7A \cos 2A} = \frac{\sin 4A}{\cos 2A}$$

This expression can be further simplified, via the double-angle formula for sine: $\sin 4A = 2 \sin 2A \cos 2A$.

$$\frac{\sin 4A}{\cos 2A} = \frac{2 \sin 2A \cos 2A}{\cos 2A} = 2 \sin 2A$$

So the answer is $\boxed{2 \sin 2A}$.

4. Prove that $\sin 20^\circ \sin 40^\circ \sin 60^\circ \sin 80^\circ = \frac{3}{16}$

Solution: We will apply the product-sum identities several times. First for $\sin 20^\circ \sin 40^\circ$ and for $\sin 60^\circ \sin 80^\circ$.

$$\begin{aligned}\cos 20^\circ &= \cos(40^\circ - 20^\circ) = \cos 40^\circ \cos 20^\circ + \sin 40^\circ \sin 20^\circ \\ \cos 60^\circ &= \cos(40^\circ + 20^\circ) = \cos 40^\circ \cos 20^\circ - \sin 40^\circ \sin 20^\circ\end{aligned}$$

If we subtract the second equation from the first one, we get that $\cos 20^\circ - \cos 60^\circ = 2 \sin 40^\circ \sin 20^\circ$

and so $\sin 40^\circ \sin 20^\circ = \frac{1}{2} (\cos 20^\circ - \cos 60^\circ)$

Similarly,

$$\begin{aligned}\cos 20^\circ &= \cos(80^\circ - 60^\circ) = \cos 80^\circ \cos 60^\circ + \sin 80^\circ \sin 60^\circ \\ \cos 140^\circ &= \cos(80^\circ + 60^\circ) = \cos 80^\circ \cos 60^\circ - \sin 80^\circ \sin 60^\circ\end{aligned}$$

If we subtract the second equation from the first one, we get that $\cos 20^\circ - \cos 140^\circ = 2 \sin 60^\circ \sin 80^\circ$

and so $\sin 60^\circ \sin 80^\circ = \frac{1}{2} (\cos 20^\circ - \cos 140^\circ)$

Let us re-write $\cos 140^\circ$ as $-\cos 40^\circ$ and then we have that $\sin 60^\circ \sin 80^\circ = \frac{1}{2} (\cos 20^\circ + \cos 40^\circ)$

$$\begin{aligned}\sin 20^\circ \sin 40^\circ \sin 60^\circ \sin 80^\circ &= \frac{1}{2} (\cos 20^\circ - \cos 60^\circ) \frac{1}{2} (\cos 20^\circ + \cos 40^\circ) \\ &= \frac{1}{4} (\cos 20^\circ - \cos 60^\circ) (\cos 20^\circ + \cos 40^\circ) \\ &= \frac{1}{4} (\cos^2 20^\circ + \cos 20^\circ \cos 40^\circ - \cos 20^\circ \cos 60^\circ - \cos 40^\circ \cos 60^\circ)\end{aligned}$$

Now we will apply the product-sum formulas again, except for the first term, $\cos^2 20^\circ$ which we will handle by the double angle formula.

$\cos 40^\circ = 2 \cos^2 20^\circ - 1$ and so $\cos^2 20^\circ = \frac{1}{2} (\cos 40^\circ + 1)$

Now we re-write $\cos 20^\circ \cos 40^\circ$

$$\begin{aligned}\cos 20^\circ &= \cos(40^\circ - 20^\circ) = \cos 40^\circ \cos 20^\circ + \sin 40^\circ \sin 20^\circ \\ \cos 60^\circ &= \cos(40^\circ + 20^\circ) = \cos 40^\circ \cos 20^\circ - \sin 40^\circ \sin 20^\circ\end{aligned}$$

If we add the two equations, we get that $\cos 20^\circ + \cos 60^\circ = 2 \cos 40^\circ \cos 20^\circ$

and so $\cos 20^\circ \cos 40^\circ = \frac{1}{2} (\cos 20^\circ + \cos 60^\circ)$

Similarly, we can re-write $\cos 20^\circ \cos 60^\circ$ as $\frac{1}{2} (\cos 40^\circ + \cos 80^\circ)$ and $\cos 40^\circ \cos 60^\circ$ as $\frac{1}{2} (\cos 20^\circ + \cos 100^\circ)$

So we have that $\sin 20^\circ \sin 40^\circ \sin 60^\circ \sin 80^\circ =$

$$\begin{aligned} &= \frac{1}{4} (\cos^2 20^\circ + \cos 20^\circ \cos 40^\circ - \cos 20^\circ \cos 60^\circ - \cos 40^\circ \cos 60^\circ) \\ &= \frac{1}{4} \left(\frac{1}{2} (1 + \cos 40^\circ) + \frac{1}{2} (\cos 20^\circ + \cos 60^\circ) - \frac{1}{2} (\cos 40^\circ + \cos 80^\circ) - \frac{1}{2} (\cos 20^\circ + \cos 100^\circ) \right) \\ &= \frac{1}{8} (1 + \cos 40^\circ + \cos 20^\circ + \cos 60^\circ - \cos 40^\circ - \cos 80^\circ - \cos 20^\circ - \cos 100^\circ) \end{aligned}$$

We can cancel out $\cos 20^\circ$ and $\cos 40^\circ$ and have that

$$\sin 20^\circ \sin 40^\circ \sin 60^\circ \sin 80^\circ = \frac{1}{8} (1 + \cos 60^\circ - \cos 80^\circ - \cos 100^\circ)$$

We know that $\cos 60^\circ = \frac{1}{2}$ and that $\cos 100^\circ = -\cos 80^\circ$ and so

$$\sin 20^\circ \sin 40^\circ \sin 60^\circ \sin 80^\circ = \frac{1}{8} \left(1 + \frac{1}{2} - \cos 80^\circ - (-\cos 80^\circ) \right) = \frac{1}{8} \cdot \frac{3}{2} = \frac{3}{16}$$

This completes our proof.

5. Prove that $\cos 20^\circ + \cos 60^\circ + \cos 100^\circ + \cos 140^\circ = \frac{1}{2}$

Solution: We will multiply this expression by $\frac{2 \sin 20^\circ}{2 \sin 20^\circ}$. Then we will distribute the numerator and then apply the product-sum identities to re-write the products.

$$\begin{aligned} &\cos 20^\circ + \cos 60^\circ + \cos 100^\circ + \cos 140^\circ = \\ &= \frac{2 \sin 20^\circ}{2 \sin 20^\circ} (\cos 20^\circ + \cos 60^\circ + \cos 100^\circ + \cos 140^\circ) \\ &= \frac{1}{2 \sin 20^\circ} (2 \sin 20^\circ \cos 20^\circ + 2 \sin 20^\circ \cos 60^\circ + 2 \sin 20^\circ \cos 100^\circ + 2 \sin 20^\circ \cos 140^\circ) \end{aligned}$$

The first term can be simplified by the double angle formula for sine: $2 \sin 20^\circ \cos 20^\circ = \sin 40^\circ$

Now we re-write $2 \sin 20^\circ \cos 60^\circ$

$$\begin{aligned} \sin 80^\circ &= \sin (60^\circ + 20^\circ) = \sin 60^\circ \cos 20^\circ + \cos 60^\circ \sin 20^\circ \\ \sin 40^\circ &= \sin (60^\circ - 20^\circ) = \sin 60^\circ \cos 20^\circ - \cos 60^\circ \sin 20^\circ \end{aligned}$$

Let us subtract the second equation from the first and get $\sin 80^\circ - \sin 40^\circ = 2 \cos 60^\circ \sin 20^\circ$

Similarly, we can re-write $2 \cos 100^\circ \sin 20^\circ$ as $\sin 120^\circ - \sin 80^\circ$ and $2 \cos 140^\circ \sin 20^\circ$ as $\sin 160^\circ - \sin 120^\circ$

So then we will have that

$$\begin{aligned} &\frac{1}{2 \sin 20^\circ} (2 \sin 20^\circ \cos 20^\circ + 2 \sin 20^\circ \cos 60^\circ + 2 \sin 20^\circ \cos 100^\circ + 2 \sin 20^\circ \cos 140^\circ) \\ &= \frac{1}{2 \sin 20^\circ} (\sin 40^\circ + \sin 80^\circ - \sin 40^\circ + \sin 120^\circ - \sin 80^\circ + \sin 160^\circ - \sin 120^\circ) \end{aligned}$$

There are cancellations: $\sin 40^\circ$, $\sin 80^\circ$, and $\sin 120^\circ$ will be canceled out and we are left with $\frac{1}{2 \sin 20^\circ} (\sin 160^\circ)$

Since they are supplements, 20° and 160° have the same sine, and so $\frac{1}{2 \sin 20^\circ} (\sin 160^\circ) = \frac{\sin 20^\circ}{2 \sin 20^\circ} = \frac{1}{2}$.

This completes our proof.

For more documents like this, visit our page at <https://teaching.martahidegkuti.com> and click on Lecture Notes. E-mail questions or comments to mhidegkuti@ccc.edu.