Sample Problems

1. Solve the triangle. $a = 3$, $b = 7$, $c = 6$

2. Consider the triangle with sides 7, 9, and 12 units long. Let α, β, and γ denote the angles of the triangle. Compute each of the following.

   a) $\cos(\alpha + \beta + \gamma)$ b) $\cos\alpha + \cos\beta + \cos\gamma$ c) the exact value of the area of the triangle.

3. In triangle ABC, $\gamma = 90°$ and $\cos\alpha = \dfrac{4}{5}$. Let D be the midpoint of side AC

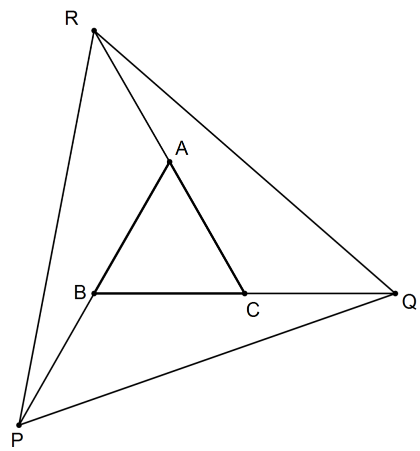
   a) Find the exact value of the cosine of angle CDB.

   b) Compute the exact value of the cosine of angle ADB.

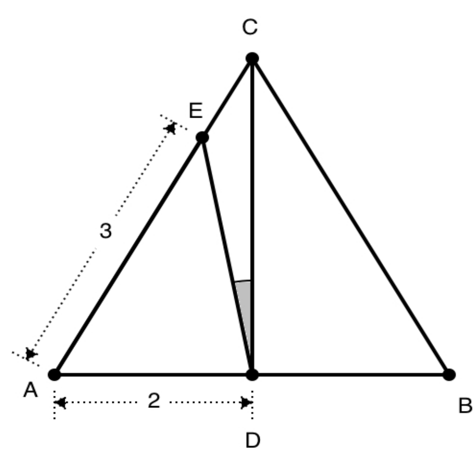
4. Point D is on side AB of triangle ABC, with $\angle ACD = \angle BCD = 60°$, $AC = 5$, and $BC = 15$. Find the length of line segment CD.

5. A triangle has sides of length a, b, and c, which are consecutive integers in increasing order, and $\cos\gamma = \dfrac{5}{16}$. Find $\cos\alpha$.

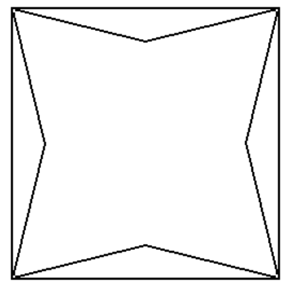
6. Suppose that ABC is a equilateral triangle with all sides of length 1 unit. We extend side AB by 1 unit beyond B to get P, we extend side BC by 1 unit beyond C to get Q and side AC beyond A to get R. Compute the length of the sides in triangle PQR.



7. All three sides of triangle ABC are 4 units long. Compute the exact value of the cosine of the angle shaded on the picture.

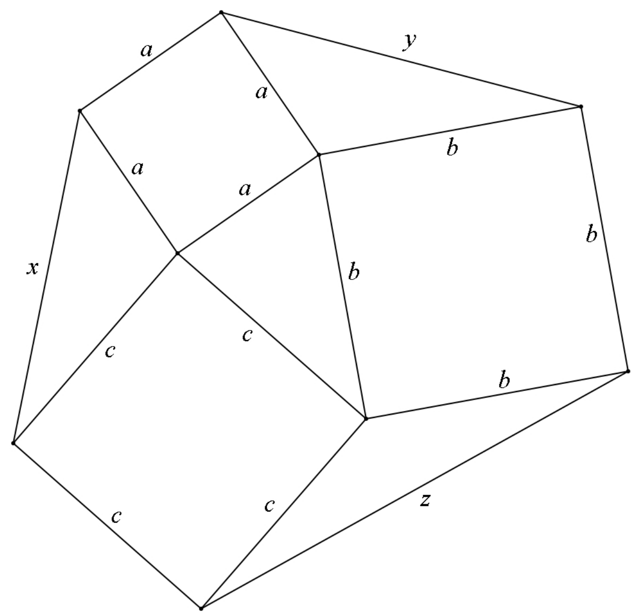


8. Consider a square with sides 1 unit long. To the inside of each side, we draw an isosceles triangle with its greatest angle, opposite the unit long base, measures 150°. Consider all vertices of these triangles that are not on the square. If we connect these vertices, we obtain a square. Compute the exact value of the area of this square.



9. A trapezoid's parallel sides are 18 and 24 units long. Another side is 15 units long, and the angle formed between this side and the longer parallel side is 74.5°. Compute the fourth side of the trapezoid and its angles.

10. Let ABC be a triangle with sides a, b, and c. We draw squares on all three sides. The vertices of the squares that are not on the triangle form a hexagon. Label the sides of the hexagon that are not on the square by x, y, and z. Prove that $x^2 + y^2 + z^2 = 3\left(a^2 + b^2 + c^2\right)$.



Answers for Sample Problems

1.) $\alpha \approx 25.208\,7655°$, $\beta \approx 96.379\,37°$, $\gamma \approx 58.411\,8645°$ 2.) a) -1 b) $\frac{37}{27}$ c) $14\sqrt{5}$

3.) a) $\frac{2\sqrt{13}}{13}$ b) $-\frac{2\sqrt{13}}{13}$ 4.) $\frac{15}{4} = 3.75$ 5.) $\frac{13}{20}$ 6.) $\sqrt{7}$ unit 7.) $\frac{3\sqrt{21}}{14}$ 8.) $2 - \sqrt{3}$

9.) 14.591 units, 105.5°, 97.84440°, 82.1556° 10.) see solutions

Solutions of Sample Problems

1. Solve the triangle. $a = 3$, $b = 7$, $c = 6$

   When using the law of cosines, we will first find the largest angle in the triangle. This way, when we next apply the law of sines (the law of cosines is also an option), we know that we are solving for angles that MUST be acute.

   First we will use the law of cosines to find β.

$$\begin{aligned} b^2 &= a^2 + c^2 - 2ac\cos\beta \\ 2ac\cos\beta &= a^2 + c^2 - b^2 \\ \cos\beta &= \frac{a^2 + c^2 - b^2}{2ac} = \frac{3^2 + 6^2 - 7^2}{2 \cdot 3 \cdot 6} = -\frac{1}{9} \end{aligned}$$

   Then $\beta = \cos^{-1}\left(-\frac{1}{9}\right) \approx 96.379\,37°$

   We can now find γ by using either the law of sines or the law of cosines. We will use the law of cosines again.

$$\begin{aligned} c^2 &= a^2 + b^2 - 2ab\cos\gamma \\ 2ab\cos\gamma &= a^2 + b^2 - c^2 \\ \cos\gamma &= \frac{a^2 + b^2 - c^2}{2ab} = \frac{3^2 + 7^2 - 6^2}{2 \cdot 3 \cdot 7} = \frac{11}{21} \end{aligned}$$

   Thus $\gamma = \cos^{-1}\left(\frac{11}{21}\right) \approx 58.411\,8645°$

   The third angle can be easily found.

$$\alpha = 180° - (\beta + \gamma) \approx 180° - (96.379\,37° + 58.411\,8645°) = 25.208\,7655°$$

   And so the answer is $\alpha \approx 25.208\,7655°$, $\beta \approx 96.379\,37°$, and $\gamma \approx 58.411\,8645°$.

2. Consider the triangle with sides 7, 9, and 12 units long. Let α, β, and γ denote the angles of the triangle. Compute each of the following.

   a) $\cos(\alpha + \beta + \gamma) = \cos 180° = -1$

   b) $\cos\alpha + \cos\beta + \cos\gamma$

Solution: Let us label the sides $a = 7$, $b = 9$, and $c = 12$. We first use the law of cosines to compute $\cos\alpha$

$$\begin{aligned} a^2 &= b^2 + c^2 - 2bc\cos\alpha \\ 2bc\cos\alpha &= b^2 + c^2 - a^2 \\ \cos\alpha &= \frac{b^2 + c^2 - a^2}{2ac} = \frac{9^2 + 12^2 - 7^2}{2 \cdot 9 \cdot 12} = \frac{22}{27} \end{aligned}$$

We compute β similarly.

$$\begin{aligned} b^2 &= a^2 + c^2 - 2ac\cos\beta \\ 2ac\cos\beta &= a^2 + c^2 - b^2 \\ \cos\beta &= \frac{a^2 + c^2 - b^2}{2ac} = \frac{7^2 + 12^2 - 9^2}{2 \cdot 7 \cdot 12} = \frac{2}{3} \end{aligned}$$

We will again use the law of cosines, this time to compute γ.

$$\begin{aligned} c^2 &= a^2 + b^2 - 2ab\cos\gamma \\ 2ab\cos\gamma &= a^2 + b^2 - c^2 \\ \cos\gamma &= \frac{a^2 + b^2 - c^2}{2ab} = \frac{7^2 + 9^2 - 12^2}{2 \cdot 7 \cdot 9} = -\frac{1}{9} \end{aligned}$$

Now $\cos\alpha + \cos\beta + \cos\gamma = \frac{22}{27} + \frac{2}{3} - \frac{1}{9} = \frac{37}{27}$

c) the exact value of the area of the triangle.

Solution: We will use the formula $A = \frac{1}{2}ab\sin\gamma$. We first need to compute $\sin\gamma$. We know that $\sin\gamma$ is positive because γ is an angle in a triangle, and so $0 < \gamma < 180°$.

$$\sin\gamma = \sqrt{1 - \cos^2\gamma} = \sqrt{1 - \left(-\frac{1}{9}\right)^2} = \sqrt{1 - \frac{1}{81}} = \sqrt{\frac{80}{81}} = \frac{4\sqrt{5}}{9}$$

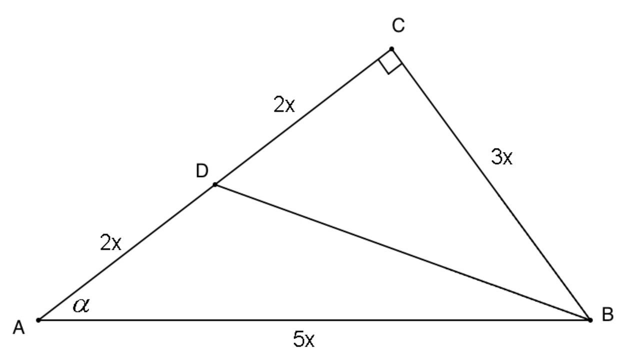
Now we can compute the exact value of the area of the triangle.

$$A = \frac{1}{2}ab\sin\gamma = \frac{1}{2} \cdot 7 \cdot 9 \left(\frac{4\sqrt{5}}{9}\right) = 14\sqrt{5}$$

3. In triangle ABC, $\gamma = 90°$ and $\cos\alpha = \frac{4}{5}$. Let D be the midpoint of side AC

a) Find the exact value of the cosine of angle CDB.

Solution: It is part of the problem to come up with a picture that depicts the data. Sometimes that is not easy and takes several attempts. However, a good picture is often an essential part of solving a geometry problem.



From the fact that this is right triangle and $\cos\alpha = \frac{4}{5}$, we conclude that this triangle is similar to the one with sides 3, 4, and 5 units and label the sides as $3x$, $4x$, and $5x$ where x is a positive number. Since D is the midpoint of line segment AC, both AD and DC are of length $2x$.

We compute the length of line segment BD using the Pythagorean Theorem:

$$BD = \sqrt{(2x)^2 + (3x)^2} = \sqrt{4x^2 + 9x^2} = \sqrt{13x^2} = \sqrt{13}x$$

We are now ready to find the cosine of angle CBD

$$\cos(\angle CDB) = \frac{2x}{\sqrt{13}x} = \frac{2}{\sqrt{13}} = \frac{2\sqrt{13}}{13}$$

b) Compute the exact value of the cosine of angle ADB.

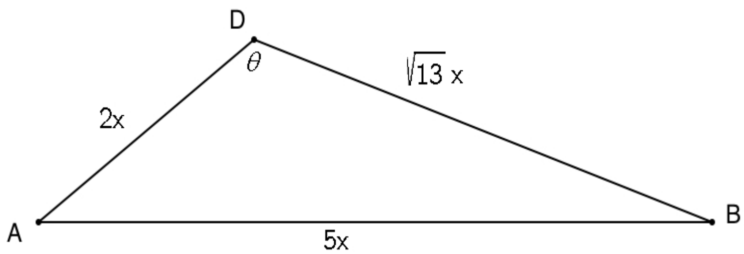
Solution 1: Angles CDB and ADB are supplements, they add up to $180°$. For all angles α,

$$\cos(180° - \alpha) = -\cos\alpha$$

and so

$$\cos(\angle ADB) = -\cos(\angle ADB) = -\frac{2\sqrt{13}}{13}$$

Solution 2: Consider triangle ABD.

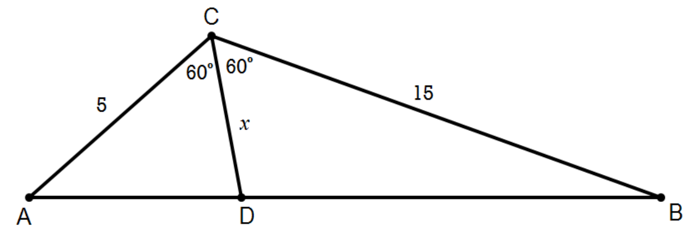


We can compute the cosine of $\theta = \angle ADB$ by stating the law of cosines for triangle ABD.

$$\begin{aligned}
(AB)^2 &= (AD)^2 + (DB)^2 - 2(AD)(DB)\cos\theta \\
(5x)^2 &= (2x)^2 + \left(\sqrt{13}x\right)^2 - 2(2x)\left(\sqrt{13}x\right)\cos\theta \\
25x^2 &= 4x^2 + 13x^2 - 4\sqrt{13}x^2\cos\theta \\
25x^2 &= \left(17 - 4\sqrt{13}\cos\theta\right)x^2 \qquad \text{divide by } x^2 \\
25 &= 17 - 4\sqrt{13}\cos\theta \\
8 &= -4\sqrt{13}\cos\theta \\
\frac{8}{-4\sqrt{13}} &= \cos\theta \\
\cos\theta &= -\frac{2}{\sqrt{13}} = -\frac{2\sqrt{13}}{13}
\end{aligned}$$

4. Point D is on side AB of triangle ABC, with $\angle ACD = \angle BCD = 60°$, $AC = 5$, and $BC = 15$. Find the length of line segment CD.

Solution: It is part of the problem to come up with a picture that depicts the data. Sometimes that is not easy and takes several attempts. However, a good picture is often an essential part of solving a geometry problem.



Let us denote line segment CD by x. Recall the formula $A = \frac{1}{2}ab\sin\gamma$. We will solve this problem by expressing the area of this triangle in two different ways.

$$\begin{aligned}
A_{\text{Triangle } ABC} &= A_{\text{Triangle } ADC} + A_{\text{Triangle } BCD} \\
\frac{1}{2}(5)(15)\sin 120° &= \frac{1}{2}(5)(x)\sin 60° + \frac{1}{2}(15)(x)\sin 60° && \text{multiply by 2} \\
75\sin 120° &= 5x\sin 60° + 15x\sin 60° \\
75\sin 120° &= 20x\sin 60° \\
75\frac{\sqrt{3}}{2} &= 20x\left(\frac{\sqrt{3}}{2}\right) && \text{divide by } \frac{\sqrt{3}}{2} \\
75 &= 20x \\
x &= \frac{75}{20} = \frac{15}{4} = 3.75
\end{aligned}$$

5. A triangle has sides of length a, b, and c, which are consecutive integers in increasing order, and $\cos\gamma = \frac{5}{16}$. Find $\cos\alpha$.

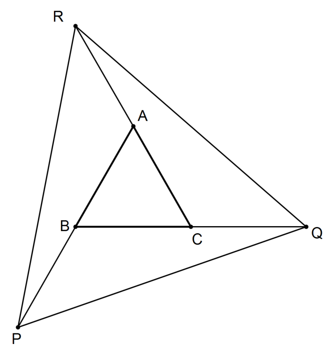
Solution: First we would want to express the fact that the three sides are consecutive integers. The usual labeling, $a = x$, $b = x+1$, and $c = x+2$ would work, but we will chose a smarter labeling that will cut down on the computations, that is $a = x-1$, $b = x$, and $c = x+1$. We will need to remember that x must be a positive integer. Since $\cos\gamma$ is given, we will state the law of cosines involving $\cos\gamma$:

$$\begin{aligned}
c^2 &= a^2 + b^2 - 2ab\cos\gamma \\
(x+1)^2 &= (x-1)^2 + x^2 - 2(x-1)x\left(\frac{5}{16}\right) \\
x^2 + 2x + 1 &= x^2 - 2x + 1 + x^2 - \frac{10}{16}\left(x^2 - x\right) \\
4x &= x^2 - \frac{5}{8}\left(x^2 - x\right) && \text{multiply by 8} \\
32x &= 8x^2 - 5\left(x^2 - x\right) \\
32x &= 8x^2 - 5x^2 + 5x \\
27x &= 3x^2 \\
0 &= 3x^2 - 27x \\
0 &= 3x(x-9) \quad \Longrightarrow \quad x_1 = 0, \quad x_2 = 9
\end{aligned}$$

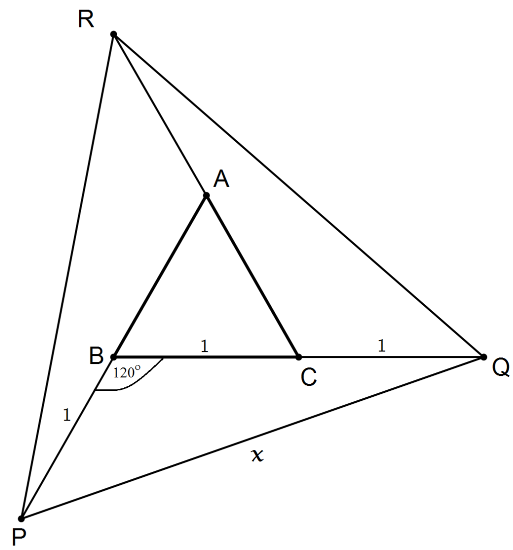
Since x can not be zero, $x = 9$ is the only solution. Because we labeled b by x, this means that the triangle's three sides are $a = 8$, $b = 9$, and $c = 10$ units. We will compute $\cos\alpha$ using the law of cosines again:

$$\begin{aligned} a^2 &= b^2 + c^2 - 2bc\cos\alpha \\ 2bc\cos\alpha &= b^2 + c^2 - a^2 \\ \cos\alpha &= \frac{b^2+c^2-a^2}{2bc} = \frac{9^2+10^2-8^2}{2\cdot 9\cdot 10} = \frac{13}{20} \end{aligned}$$

6. Suppose that ABC is a equilateral triangle with all sides of length 1 unit. We extend side AB by 1 unit beyond B to get P, we extend side BC by 1 unit beyond C to get Q and side AC beyond A to get R. Compute the length of the sides in triangle PQR.



Solution: By symmetry, triangle PQR is also equilateral and so we just need to compute the length of any side. Consider triangle PBQ.

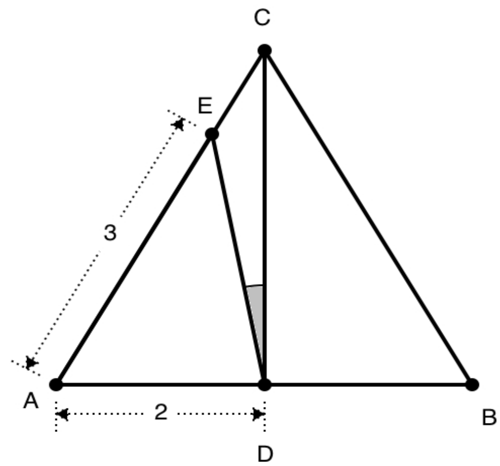


Side BP is 1 unit long and side BQ is 2 units long. Angle PBQ has measure 120° since it is the supplement of the inner angle which measures 60°. We denote side PQ by x and state the law of cosines on triangle PBQ.

$$\begin{aligned} x^2 &= 1^2 + 2^2 - 2\cdot 1\cdot 2\cdot\cos 120^\circ \qquad \cos 120^\circ = -\frac{1}{2} \\ x^2 &= 5 - 4\left(-\frac{1}{2}\right) \\ x^2 &= 5+2 \\ x^2 &= 7 \\ x &= \pm\sqrt{7} \end{aligned}$$

Since x is a distance, we discard the negative solution of the equation. The triangle PQR has sides of length $\sqrt{7}$.

7. All three sides of triangle ABC are 4 units long. Compute the exact value of the cosine of the angle shaded on the picture below.



Solution: We will solve this problem by stating the law of cosines for triangle CDE. We first need to find the length of its three sides.

First, side EC is clearly 1 unit long. We can compute the length of CD via the Pythagorean theorem

$$\begin{aligned} CD^2 + 2^2 &= 4^2 \\ CD^2 &= 12 \quad \Longrightarrow \quad CD = \sqrt{12} \end{aligned}$$

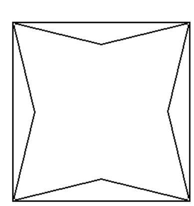
We will now use the law of cosines in triangle ADE to compute the length of ED.

$$\begin{aligned} ED^2 &= 2^2 + 3^2 - 2 \cdot 2 \cdot 3 \cdot \cos 60^\circ \\ ED^2 &= 4 + 9 - 12 \cdot \frac{1}{2} = 13 - 6 = 7 \quad \Longrightarrow \quad ED = \sqrt{7} \end{aligned}$$

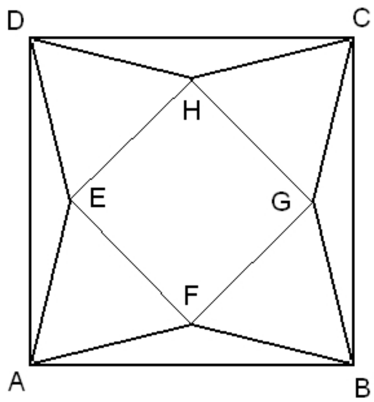
So now we know that $EC = 1$, $CD = \sqrt{12}$, and $ED = \sqrt{7}$. We need to compute the cosine of the angle oposite side EC. Let us enote this angle by α.

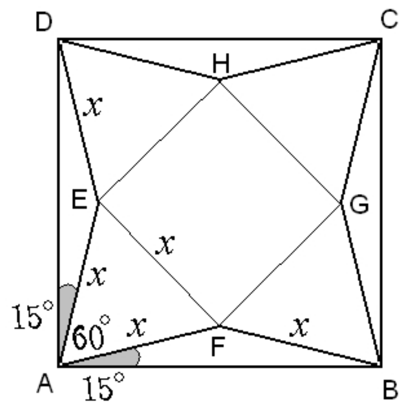
$$\begin{aligned} 1^2 &= \left(\sqrt{12}\right)^2 + \left(\sqrt{7}\right)^2 - 2\sqrt{12}\sqrt{7}\cos\alpha \\ 1 &= 19 - 2\sqrt{84}\cos\alpha \\ 2\sqrt{84}\cos\alpha &= 18 \\ \cos\alpha &= \frac{18}{2\sqrt{84}} = \frac{9}{\sqrt{84}} = \frac{9\sqrt{84}}{84} = \frac{9\sqrt{84}}{84} = \frac{3\sqrt{84}}{28} = \frac{3 \cdot 2\sqrt{21}}{28} = \frac{3\sqrt{21}}{14} \end{aligned}$$

8. Consider a square with sides 1 unit long. To the inside of each side, we draw an isosceles triangle with its greatest angle, opposite the unit long base, measures 150°. Consider all vertices of these triangles that are not on the square. If we connect these vertices, we obtain a square. Compute the exact value of the area of this square.



Solution: This problem can be solved without the law of cosines, but it is much easier using it. Let us first connect the four points, denoted by E, F, G, and H as shown on the picture below.

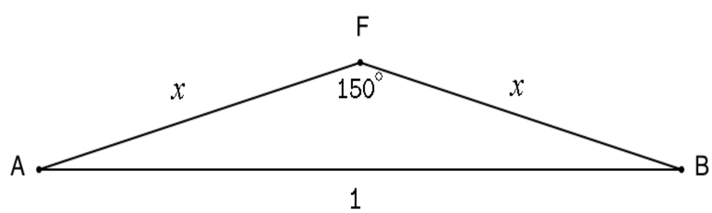




Consider first triangle ABF. By symmetry, $AF = BF$ and so the triangle is isosceles. Since angle $AFB = 150°$, the angles FAB and ABF must measure $15°$. By symmetry, angle EAD must also measure $15°$.

Consider now angle EAF. It must measure $90° - (15° + 15°) = 60°$. Consider triangle AEF. By symmetry, $AE = EF$ and so the angles opposite those sides are also the same. Since the third angle, $\angle EAF = 60°$, the other two angles must add up to $120°$. Because they are equal, they must measure $60°$ and so the triangle is equilateral and $AE = EF = AF$.

The area of the square is x^2 and we can find its value by stating the law of cosines on triangle ABF.



$$\begin{aligned} x^2 + x^2 - 2(x)(x)\cos 150° &= 1^2 \\ 2x^2 - 2x^2\left(-\frac{\sqrt{3}}{2}\right) &= 1 \\ x^2\left(2+\sqrt{3}\right) &= 1 \\ x^2 &= \frac{1}{2+\sqrt{3}} \end{aligned}$$

We rationalize our value for x^2.

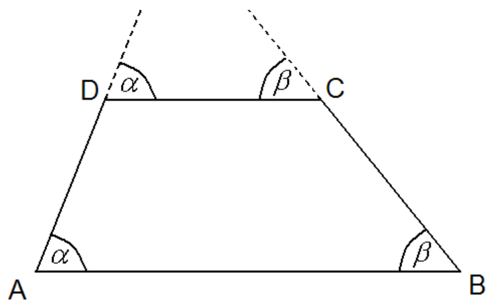
$$x^2 = \frac{1}{2+\sqrt{3}} = \frac{1}{2+\sqrt{3}} \cdot \frac{2-\sqrt{3}}{2-\sqrt{3}} = \frac{2-\sqrt{3}}{1} = 2-\sqrt{3}$$

This is the area of the square.

9. A trapezoid's parallel sides are 18 and 24 units long. Another side is 15 units long, and the angle formed between this side and the longer parallel side is 74.5°. Compute the fourth side of the trapezoid and its angles.

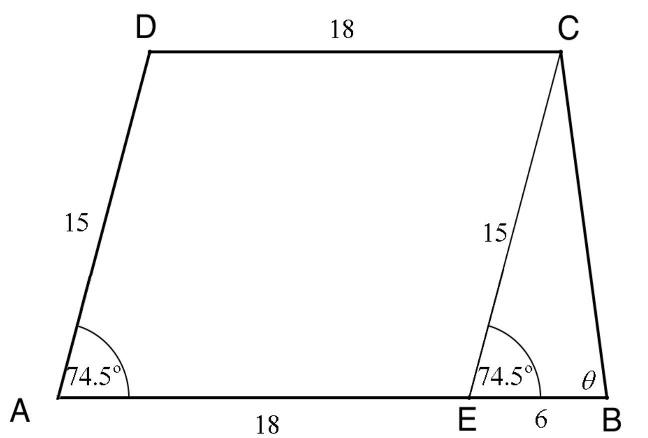
Solution: We will first prove a fact that is true for all trapezoids. The two angles formed at a trapezoid's side conntecting parallel sides are supplements, they add up to 180°.

proof: Consider the picture below. We denoted the angle by A by α and the angle by B by β. Then we extended line AD beyond D and line BC beyond C.



These new angles formed are again α and β because sides AB and CD are parallel. Now it is easy to see that $\alpha + \angle ADC = 180°$ and $\beta + \angle BCD = 180°$.

The solution of a geometry problem is often very easy after a well chosen line was drawn in. Consider the picture below.



Let us draw a line through C that is parallel to side AC. Now $AECD$ is a parallelogram, because it has two pairs of parallel sides. It is a propoerty of parallelograms that opposite sides have the same length. Thus $AE = 18$ and consequently, $EB = 24 - 18 = 6$ units. Also, angle $CEB = 74.5°$ because AD and CE are parallel.

Our smart line reduced the problem considerably. We can use the law of cosines to compute side BC:

$$\begin{aligned} BC^2 &= 15^2 + 6^2 - 2 \cdot 15 \cdot 6 \cos 74.5° \\ BC^2 &\approx 212.897\,092\,306 \quad \Longrightarrow \quad BC \approx 14.590994 \text{ unit} \end{aligned}$$

We can use the law of sines again to compute angle EBC (denoted by θ)

$$\begin{aligned}
15^2 &= 6^2 + BC^2 - 2 \cdot 6 \cdot BC \cos\theta \\
12BC\cos\theta &= 6^2 + BC^2 - 15^2 \\
\cos\theta &= \frac{6^2 + BC^2 - 15^2}{12BC} \approx \frac{6^2 + (14.590994)^2 - 15^2}{12\,(14.590994)} \approx 0.1364832 \\
\theta &\approx \cos^{-1}(0.1364832) \approx 82.1556^\circ
\end{aligned}$$

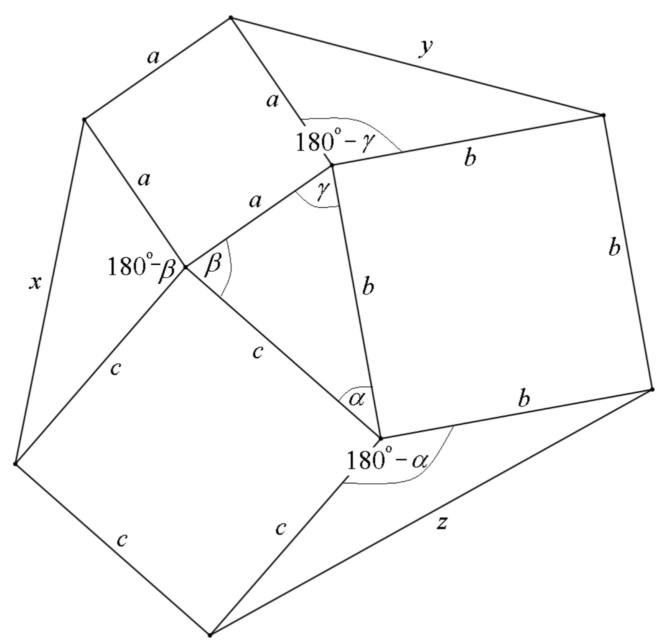
The other two angles can be found easily by using the fact that the angles along a side connecting parallel sides are supplemental.

$$\angle ADC = 180^\circ - 74.5^\circ = 105.5^\circ \quad \text{and} \quad \angle BCD = 180^\circ - \theta = 180^\circ - 82.1556^\circ = 97.8444^\circ$$

Thus the missing side and angles are: 14.591 units, 105.5°, 97.84440°, and 82.1556°.

10. Let ABC be a triangle with sides a, b, and c. We draw squares on all three sides. The vertices of the squares that are not on the triangle form a hexagon. Label the sides of the hexagon that are not on the square by x, y, and z. Prove that $x^2 + y^2 + z^2 = 3\left(a^2 + b^2 + c^2\right)$.

proof: Consider the picture below.



We first establish that the angles, shown on the picture are complements to α, β, and γ, correspondingly. Also, recall that for all angles θ, $\cos(180^\circ - \theta) = -\cos\theta$. Let us state the law of cosines for the triangles that include x, y, and z as a side.

$$x^2 = a^2 + c^2 - 2ac\cos(180^\circ - \beta) \qquad y^2 = a^2 + b^2 - 2ab\cos(180^\circ - \gamma)$$
$$x^2 = a^2 + c^2 + 2ac\cos\beta \qquad y^2 = a^2 + b^2 + 2ab\cos\gamma$$

and

$$\begin{aligned}
z^2 &= b^2 + c^2 - 2bc\cos(180^\circ - \alpha) \\
z^2 &= b^2 + c^2 + 2bc\cos\alpha
\end{aligned}$$

We add these three equations and get

$$\begin{aligned} x^2+y^2+z^2 &= a^2+c^2+2ac\cos\beta+a^2+b^2+2ab\cos\gamma+b^2+c^2+2bc\cos\alpha \\ x^2+y^2+z^2 &= 2a^2+2b^2+2c^2+2ac\cos\beta+2ab\cos\gamma+2bc\cos\alpha \end{aligned}$$

We will now "get rid" of the expressions $2ac\cos\beta$, $2ab\cos\gamma$, and $2ab\cos\alpha$ by using the law of cosines stated on the triangle abc.

$$\begin{aligned} a^2 &= b^2+c^2-2bc\cos\alpha \\ 2bc\cos\alpha &= b^2+c^2-a^2 \end{aligned} \qquad \begin{aligned} b^2 &= a^2+c^2-2ac\cos\beta \\ 2ac\cos\beta &= a^2+c^2-b^2 \end{aligned} \qquad \begin{aligned} c^2 &= a^2+b^2-2ab\cos\gamma \\ 2ab\cos\gamma &= a^2+b^2-c^2 \end{aligned}$$

We substitute these into our equation

$$\begin{aligned} x^2+y^2+z^2 &= 2a^2+2b^2+2c^2+2ac\cos\beta+2ab\cos\gamma+2bc\cos\alpha \\ &= 2a^2+2b^2+2c^2+\left(a^2+c^2-b^2\right)+\left(a^2+b^2-c^2\right)+\left(b^2+c^2-a^2\right) \\ &= 3a^2+3b^2+3c^2 \end{aligned}$$

This completes our proof.

