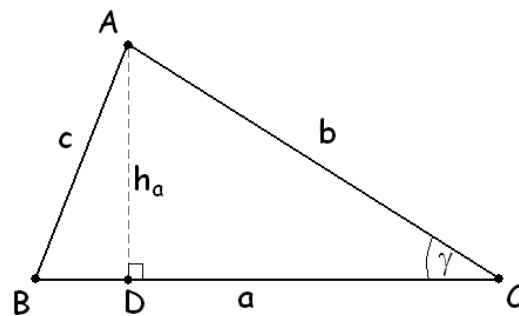


Theorem: In any triangle, $A = \frac{1}{2}ab \sin \gamma$.

$$A = \frac{1}{2}ah_a \text{ and } h_a = b \sin \gamma \text{ from triangle } ADC.$$



Theorem: (The Law of Sines)

$$\text{In any triangle, } \frac{a}{\sin \alpha} = \frac{b}{\sin \beta} = \frac{c}{\sin \gamma} = 2R$$

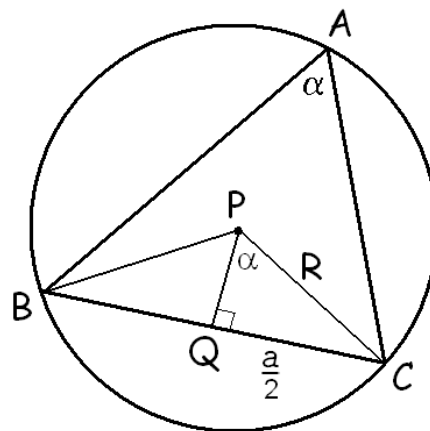
proof: Let P be the center of the circumscribed circle.

$\angle CPB = 2\alpha$ because angles CPB and CAB are central and circumscribed angles belonging to the same arc.

Triangle CPB is isosceles as $\overline{BP} = \overline{PC} = R$. Therefore, $\overline{PQ}$ bisects the angle at P and the opposite side, BC . Using right triangle trigonometry on triangle PQC , we get that $\sin \alpha = \frac{a}{2R}$.

$$\sin \alpha = \frac{a}{2R} \implies 2R \sin \alpha = a \implies 2R = \frac{a}{\sin \alpha}.$$

Similarly, $\sin \beta = \frac{b}{2R} \implies 2R = \frac{b}{\sin \beta}$ and $\sin \gamma = \frac{c}{2R} \implies 2R = \frac{c}{\sin \gamma}$. This completes our proof.



We also have a formula for the radius of the circumscribed circle: $R = \frac{a}{2 \sin \alpha} = \frac{b}{2 \sin \beta} = \frac{c}{2 \sin \gamma}$

Note that most of the law of sines can be easily derived using the sine area formula, by looking at the area of the triangle in two different ways.

$$\begin{aligned} \frac{1}{2}ab \sin \gamma &= \frac{1}{2}ac \sin \beta && \text{divide by } \frac{1}{2}a \\ b \sin \gamma &= c \sin \beta && \text{divide by } \sin \beta \sin \gamma \\ \frac{b}{\sin \beta} &= \frac{c}{\sin \gamma} \end{aligned}$$

Using this approach, we know that those ratios are equal to each other, but not that they are also equal to the diameter of the circumscribed circle.

Theorem: (The Projection Theorem) In any triangle, $a = b \cos \gamma + c \cos \beta$.

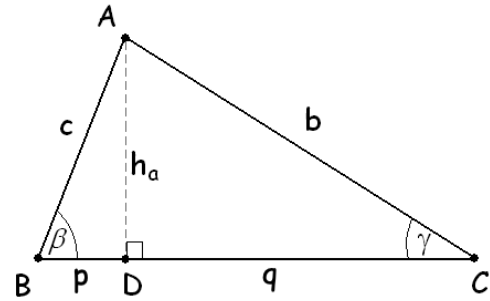
proof: Let AD be the altitude belonging to side $a = BC$.

Let $p = BD$ and $q = DC$. So $a = p + q$.

In the right triangle BDA , $\cos \beta = \frac{p}{c} \implies p = c \cos \beta$.

Similarly, in triangle DCA , $\cos \gamma = \frac{q}{b} \implies q = b \cos \gamma$.

$$a = \overline{BC} = p + q = c \cos \beta + b \cos \gamma.$$



Similar computations yield for $b = a \cos \gamma + c \cos \alpha$ and $c = a \cos \beta + b \cos \alpha$.