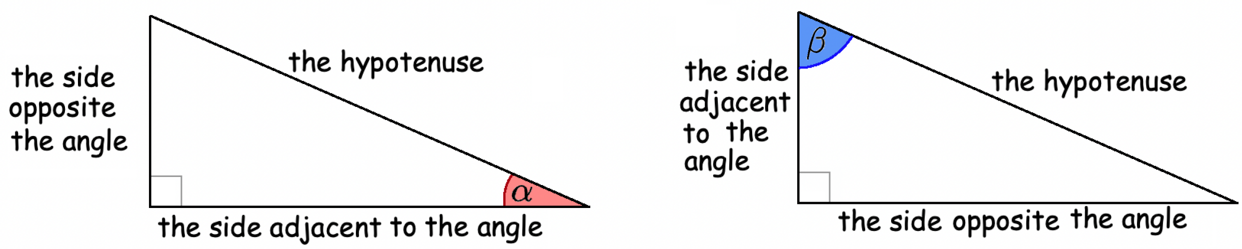
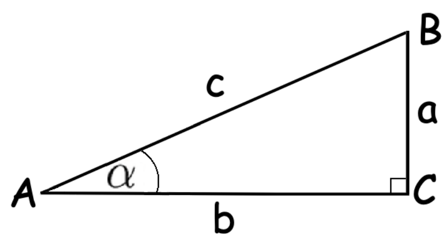
Part 1 - The Definitions

We will now define some fundamental concepts of trigonometry. Let α be an acute angle. (An angle α is acute if $0 < \alpha < 90°$).

Let us draw a right triangle that also contains α as an angle. Let us locate the angle α. The longest side, the hypotenuse is always opposite the right angle. To the other two sides we will refer to as the side opposite α and the side adjacent to α. It is important to understand that 'opposite' alone makes no sense in this context. It is opposite the angle. And if we change the location of the angle, that will result in changes in what sides we call what. Suppose that the red angle is labeled α and the blue angle is labeled by β.

The horizontal size is adjacent to α but opposite to β. The vertical side is opposite to α but adjacent to β.





Suppose we measure all three sides. The following trigonometric values belonging to angle α are defined as shows below.

Sine of α is the ratio of the lengths of two sides: the side opposite α, divided by the length of the hypotenuse.

$$\sin\alpha = \frac{\text{length of the side opposite } \alpha}{\text{length of hypotenuse}} = \frac{a}{c}$$

Cosine of α is the ratio of the lengths of two sides: the side adjacent to α, divided by the length of the hypotenuse.

$$\cos\alpha = \frac{\text{length of the side adjacent to } \alpha}{\text{length of hypotenuse}} = \frac{b}{c}$$

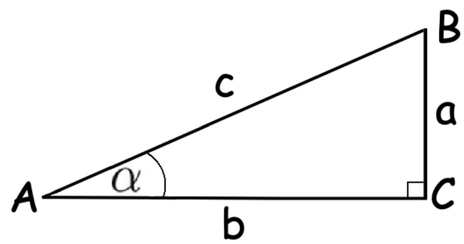
Tangent of α is the ratio of the lengths of two sides: the side opposite to α, divided by the length of the side adjacent to α.

$$\tan\alpha = \frac{\text{length of the side opposite } \alpha}{\text{length of the side adjacent to } \alpha} = \frac{a}{b}$$

These three are the most important ones, you must memorize these definitions. There are three additional definitions, but their significance will be more obvious when studying calculus. In the mean time, remember these in terms of sine, cosine, and tangent. Cosecant is the reciprocal of sine, secant is the reciprocal of cosine, and cotangent is the reciprocal of tangent.

Cosecant of α is the reciprocal of $\sin\alpha$.

$$\csc\alpha = \frac{\text{length of hypotenuse}}{\text{length of the side opposite }\alpha} = \frac{c}{a}$$



Secant of α is the reciprocal of $\cos\alpha$.

$$\sec\alpha = \frac{\text{length of hypotenuse}}{\text{length of the side adjacent to }\alpha} = \frac{c}{b}$$

Cotangent of α is the reciprocal of $\tan\alpha$.

$$\cot\alpha = \frac{\text{length of the side adjacent to }\alpha}{\text{length of the side opposite }\alpha} = \frac{b}{a}$$

Discussion:

1. The definitions are sort of vague. Do they uniquely determine these values? Suppose that an acute angle α is given. Ann and Bryne both draw a right triangle with α in it, measure the sides and compute $\sin\alpha$, $\cos\alpha$, and so on. But Ann draws a small cute triangle while Bryne draws a huge triangle. The sides are clearly different. What if $\sin\alpha$ is a different number based on Ann's triangle than $\sin\alpha$ based on Bryne's triangle? The would mean that sine of an angle is not well-defined. Well?

2. Prove that for any acute angle α, both $\sin\alpha$ and $\cos\alpha$ are positive, a real number (i.e. unit-less), and the values are always less than 1.

3. Prove that any positive number can be a tangent of an angle. Can you draw a right triangle that would lead to $\tan\alpha = 100$?

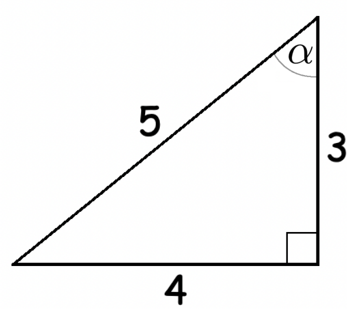
Example 1. Consider the right triangle with sides 3, 4, and 5 units long. Find all trigonometric values of the second largest angle in the triangle.

Solution: The second largest angle is opposite the second longest side, 4 units long.

$$\sin\alpha = \frac{\text{length of the side opposite }\alpha}{\text{length of hypotenuse}} = \frac{4}{5} \qquad \csc\alpha = \frac{5}{4}$$

$$\cos\alpha = \frac{\text{length of the side adjacent to }\alpha}{\text{length of hypotenuse}} = \frac{3}{5} \qquad \sec\alpha = \frac{5}{3}$$

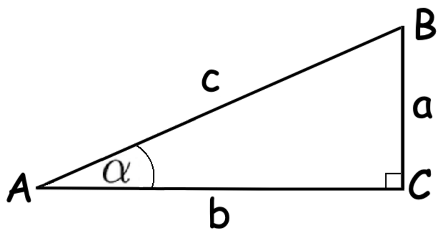
$$\tan\alpha = \frac{\text{length of the side opposite }\alpha}{\text{length of side adjacent to }\alpha} = \frac{4}{3} \qquad \cot\alpha = \frac{3}{4}$$



> **Theorem:** For any acute angle α,
> $$(\sin\alpha)^2 + (\cos\alpha)^2 = 1$$

Before we prove the statement, let us introduce some new notation. Instead of $(\sin\alpha)^2$, we will write $\sin^2\alpha$

Using the standard notation, $\sin\alpha = \frac{a}{c}$ and $\cos\alpha = \frac{b}{c}$. Then



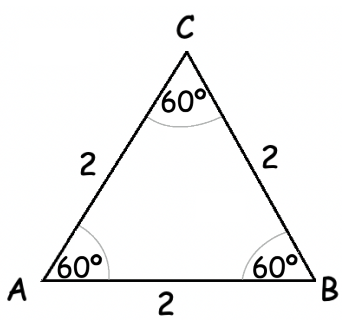
$$\begin{aligned}\sin^2\alpha + \cos^2\alpha &= \left(\frac{a}{c}\right)^2 + \left(\frac{b}{c}\right)^2 = \frac{a^2}{c^2} + \frac{b^2}{c^2} \\ &= \frac{a^2+b^2}{c^2} = \frac{c^2}{c^2} = 1\end{aligned}$$

At the core of this proof was the Pythagorean theorem. We will see that trigonometry is especially rich in identities. This one is the first of many to follow and also, probably the most fundamental identity. It is called the Pythagorean identity.

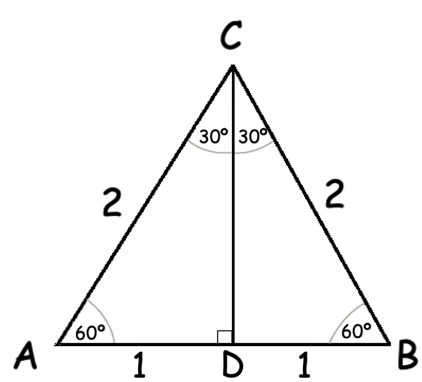
Part 2 - Exact Values

The exact value of trigonometric functions of most angles can not be determined by elementary techniques. At this point, we are to imagine mathematicians drawing right triangles and measuring sides to obtain approximate values. There are a few angles, however, that are exception to this; we can compute the exact values of trigonometric functions. Certain symmetries and the Pythagorean theorem enables us to do that, in case of $30°$, $45°$, and $60°$.

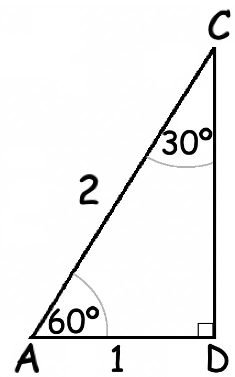
We start by drawing an equilateral triangle with sides 2 units long. All three angles of this triangle measure $60°$.



Let D be the midpoint of side AB. We connect points C and D. Because the triangle is isosceles, this line is perpendicular to the base AB and cuts the triangle into two congruent right triangles. Consequently, the two angles created at point C both measure $30°$.



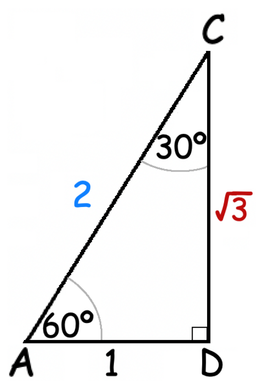
Let us now focus on just one half of the picture, triangle ADC. This triangle has angles $30°$, $60°$, and $90°$. We know that two of its sides are 1 and 2 units long. Notice that the hypotenuse is 2 units long.



We can easily compute the missing side using the Pythagorean theorem. If side BD is denoted by x, then

$$\begin{aligned} x^2+1^2 &= 2^2 \\ x^2+1 &= 4 \\ x^2 &= 3 \\ x &= \pm\sqrt{3} \quad \Longrightarrow \quad x=\sqrt{3} \end{aligned}$$

Now that we have the exact value of all three sides of the triangle, we can compute all trigonometric function values for $30°$ and $60°$ using this triangle.



Trigonometric Function Values of $30°$

$$\sin 30° = \frac{\text{opposite}}{\text{hypotenuse}} = \frac{1}{2} \qquad \csc 30° = \frac{\text{hypotenuse}}{\text{opposite}} = \frac{2}{1} = 2$$

$$\cos 30° = \frac{\text{adjacent}}{\text{hypotenuse}} = \frac{\sqrt{3}}{2} \qquad \sec 30° = \frac{\text{hypotenuse}}{\text{adjacent}} = \frac{2}{\sqrt{3}}$$

$$\tan 30° = \frac{\text{opposite}}{\text{adjacent}} = \frac{1}{\sqrt{3}} \qquad \cot 30° = \frac{\text{adjacent}}{\text{opposite}} = \frac{\sqrt{3}}{1} = \sqrt{3}$$

Trigonometric Function Values of $60°$

$$\sin 60° = \frac{\text{opposite}}{\text{hypotenuse}} = \frac{\sqrt{3}}{2} \qquad \csc 60° = \frac{\text{hypotenuse}}{\text{opposite}} = \frac{2}{\sqrt{3}}$$

$$\cos 60° = \frac{\text{adjacent}}{\text{hypotenuse}} = \frac{1}{2} \qquad \sec 60° = \frac{\text{hypotenuse}}{\text{adjacent}} = \frac{2}{1} = 2$$

$$\tan 60° = \frac{\text{opposite}}{\text{adjacent}} = \frac{\sqrt{3}}{1} = \sqrt{3} \qquad \cot 60° = \frac{\text{adjacent}}{\text{opposite}} = \frac{1}{\sqrt{3}} = \frac{1}{\sqrt{3}}$$

Note: Consider the exact value of $\tan 30°$. The ratio of the sides is $\frac{1}{\sqrt{3}}$. Sometimes this number is rationalized and is presented as $\frac{\sqrt{3}}{3}$. Here is the computation:

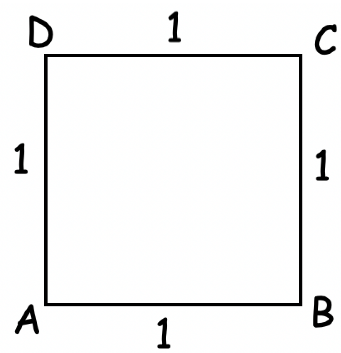
$$\frac{1}{\sqrt{3}} = \frac{1}{\sqrt{3}} \cdot 1 = \frac{1}{\sqrt{3}} \cdot \frac{\sqrt{3}}{\sqrt{3}} = \frac{\sqrt{3}}{3}$$

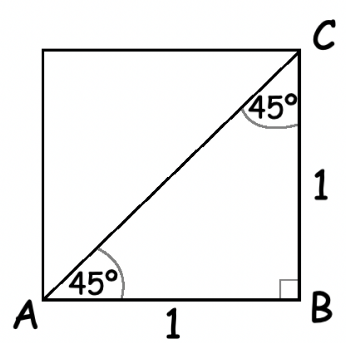
Similarly, $\csc 60^\circ = \dfrac{2}{\sqrt{3}}$ can be rationalized and presented as $\dfrac{2\sqrt{3}}{3}$. While the rationalized form is considered 'simplified', both forms have their own advantages, and it is a useful skill to know which form is better for us in different situations. For example, if we add several fractions of different denominators and $\tan 30^\circ$ is one of them, the rationalized form is much better. On the other hand, if we need to square $\tan 30^\circ$, it is easier to work with the other form:

$$(\tan 30^\circ)^2 = \left(\frac{1}{\sqrt{3}}\right)^2 = \frac{1}{3}.$$

Trigonometric Function Values of 45°

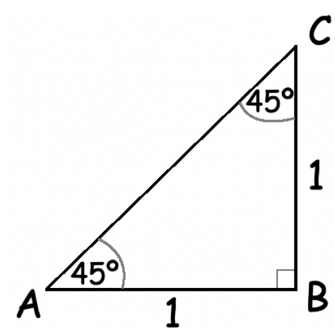
We start by drawing a square with sides 1 units long. We then draw the diagonal AC. This line cuts the square into two identical, isosceles right triangles. Consequently, the angles created at points A and C both measure 45°.





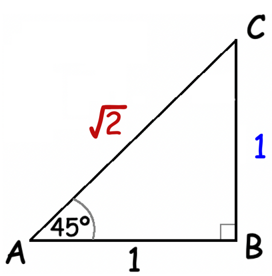
Consider now triangle ABC. We compute the hypotenuse AC by the Pythagorean theorem: if we denote AC by x, we have that

$$\begin{aligned} 1^2 + 1^2 &= x^2 \\ 2 &= x^2 \\ x &= \pm\sqrt{2} \quad \Longrightarrow \quad x = \sqrt{2} \end{aligned}$$



To avoid confusion, we mark only one of the 45° angle on the picture below. Now that we know all three sides of the triangle, we can compute all trigonometric values of 45°.

Trigonometric Function Values of 45°



$$\sin 45^\circ = \frac{\text{opposite}}{\text{hypotenuse}} = \frac{1}{\sqrt{2}} \qquad \csc 45^\circ = \frac{\text{hypotenuse}}{\text{opposite}} = \frac{\sqrt{2}}{1} = \sqrt{2}$$

$$\cos 45^\circ = \frac{\text{adjacent}}{\text{hypotenuse}} = \frac{1}{\sqrt{2}} \qquad \sec 45^\circ = \frac{\text{hypotenuse}}{\text{adjacent}} = \frac{\sqrt{2}}{1} = \sqrt{2}$$

$$\tan 45^\circ = \frac{\text{opposite}}{\text{adjacent}} = \frac{1}{1} = 1 \qquad \cot 45^\circ = \frac{\text{adjacent}}{\text{opposite}} = \frac{1}{1} = 1$$

Note: $\sin 45^\circ = \dfrac{1}{\sqrt{2}}$ can be rationalized: $\dfrac{1}{\sqrt{2}} = \dfrac{1}{\sqrt{2}} \cdot 1 = \dfrac{1}{\sqrt{2}} \cdot \dfrac{\sqrt{2}}{\sqrt{2}} = \dfrac{\sqrt{2}}{2}$. Again, while the rationalized form is considered simplified, both forms have their own advantages.

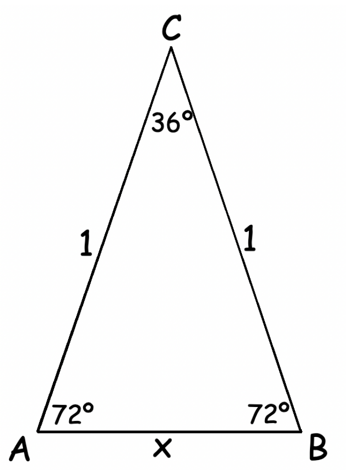
What about all other acute angles? At this point, we need to imagine a lot of triangles drawn, angles measured, and the trigonometric ratios recorded as decimals. For example, if $\alpha = 25°$, then the exact value of the sine is denoted by $\sin 25°$ and the approximate value can be found using the calculator: $\sin 25° \approx 0.4226183$.

From now on, it goes without saying that if we are dealing with a famous angle, we must use the exact values, and if the angle is not famous, we usually use the approximate values. We often don't know at the beginning of a problem which is the case.

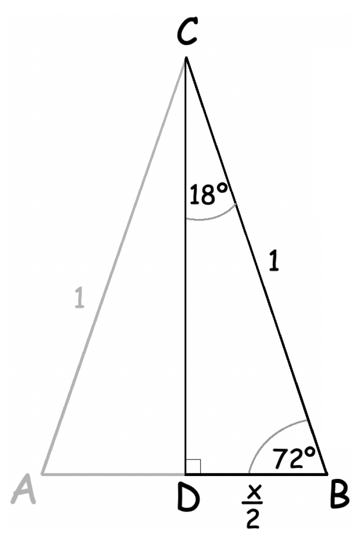
Enrichment

There are some additional angles with algebraically approachable trigonometric function values. We can also compute the exact values for the trigonometric functions of $18°$ and $72°$. These computations use the Pythagorean theorem and similar triangles and are quite interesting.

Consider an isosceles triangle with angles $72°$, $72°$, and $36°$. Label the equal sides as 1 and the third side by x.

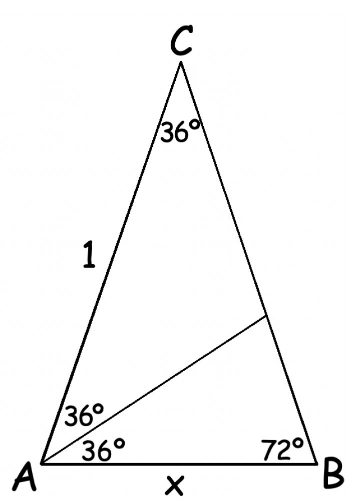


If we could find the value of x, then we could find the trigonometric values for both $18°$ and $72°$. But can we find x?



The answer is: yes! We can find the value of x. Let us draw the angle bisector at point A. The angle bisector splits the triangle into two isosceles triangles, one of which is similar to the original triangle. Use this to set up and solve an equation for x.

Use your result to compute the exact values of $\sin 18°$ and $\sin 72°$.



Part 3 - Applications

Right triangle trigonometry can be thought of as a bridge between sides and angles. There are many applications in physics, engineering, architecture, geography, even in computer science.

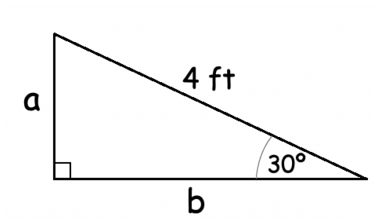
Again, if the angle is famous, we must use exact values. Otherwise, we will use approximate values. This distinction goes without saying in trigonometry and is demonstrated in the next example.

Example 2. Find the exact and approximate value of the shorter sides of a right triangle with hypotenuse 4 ft long and smallest angle

a) $30°$ b) $37°$

Solution: a) $30°$ is a famous angle. We must know the exact values of all trigonometric functions. The recommendation is that as long as we need to, we draw the semi-regular triangle (actually similar to this triangle), apply the Pythagorean theorem and find the ratios as described in Part 2.

We can argue that the data given uniquely determines the triangle. If we know it's a right triangle with a $30°$ angle in it, we know all three angles. The three angles uniquely determine a triangle up to similarity. We can now enlarge the triangle until we get the one with the 4 ft long hypotenuse. This means that we can find everything about this triangle.



We will state simple trigonometric ratios and then solve for the unknown we need.

$$\sin 30° = \frac{a}{4 \text{ ft}} \implies a = (4 \text{ ft}) \cdot \sin 30° = (4 \text{ ft}) \cdot \frac{1}{2} = 2 \text{ ft}$$

$$\cos 30° = \frac{b}{4 \text{ ft}} \implies b = (4 \text{ ft}) \cdot \cos 30° = (4 \text{ ft}) \cdot \frac{\sqrt{3}}{2} = 2\sqrt{3} \text{ ft}$$

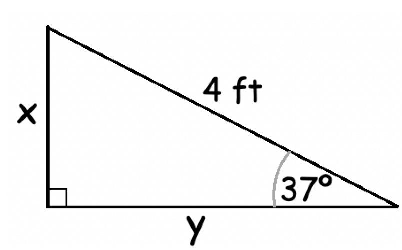
So the exact values are: $a = 2$ ft and $b = 2\sqrt{3}$ ft

and the approximate values are: $a = 2$ ft and $b \approx 3.4641$ ft

b) The question is very similar, only this time the angle is not famous.

$$\sin 37° = \frac{x}{4 \text{ ft}} \implies x = (4 \text{ ft}) \cdot \sin 37° \approx 2.40726 \text{ ft}$$

$$\cos 37° = \frac{y}{4 \text{ ft}} \implies y = (4 \text{ ft}) \cdot \cos 37° \approx 3.1945 \text{ ft}$$



So the exact values are: $x = 4 \sin 37°$ ft and $y = 4 \cos 37°$ ft

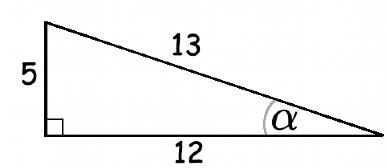
and the approximate values are: $x \approx 2.40726$ ft and $y \approx 3.1945$ ft

We can use the calculator to find the ratio that belongs to the angle. The calculator can also find the angle, given the ratio. For example, if we want to find the acute angle whose tangent is $\frac{1}{2}$, we just enter $\tan^{-1}\left(\frac{1}{2}\right)$. Angles can be measured in several ways, so we need to make sure that the calculator is set for degree measure of angles. The calculator will give us $26.565°$ as approximate value.

Example 3. Find the degree measure of the smallest angle in the right triangle with sides 5, 12, and 13 units long.

Solution: The smallest angle is opposite the shortest side.

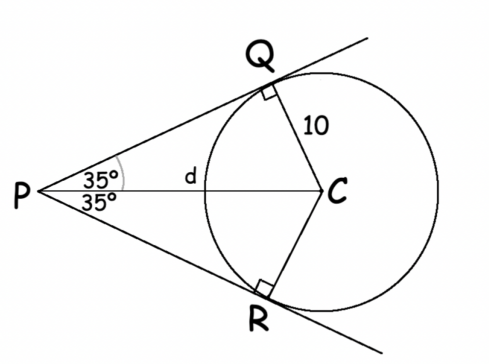
We need to ask the calculator to give us the angle whose sine is $\frac{5}{13}$. We do that by entering $\sin^{-1}\left(\frac{5}{13}\right)$. The answer is approximately $22.6199°$.



Example 4. Suppose that C is the center of a circle with radius 10 units. Find all the points P with the following property: the two tangent lines drawn to the circle form an angle of $70°$

Solution: Recall that the tangent line drawn to a circle is perpendicular to the radius drawn to the point of tangency.

Suppose that P is an external point P with the required property. Let us connect it with the center C. Let us label the points of tangency as Q and R. Triangles PCQ and PCR are two congruent right triangles. Focusing on just one of them PCQ, we need to find the length of $AC = d$, given that the angle at P is $35°$ and side $QC = 10$ units. We can find a trigonometric statement that connects d, $r = 10$, and $\alpha = 35°$, and that is to state $\sin\alpha$ for triangle PCQ and solve for d.



$$\sin 35° = \frac{10}{d} \quad \Longrightarrow \quad d = \frac{10}{\sin 35°} \approx 17.434468$$

Therefore, all points P with this property must be at a distance of 17.434468 units from C. Those points are on a circle centered at C and with a radius of 17.434468 units.

In architecture, the angle between the two tangent lines is referred to as the angle of view - so there are plenty of real world applications.

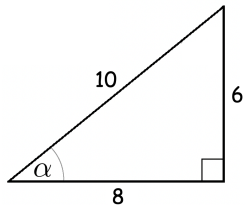
Example 5. Consider the right triangle with sides 6, 8 and 10 units. Let α denote the smallest angle in the triangle.

a) Compute $\sin\alpha$ and $\cos\alpha$.

b) Find the right triangle that is similar to the triangle with sides 6, 8, and 10 units and has a hypotenuse 1 unit long. Find $\sin\alpha$ and $\cos\alpha$ again where α is the smallest angle.

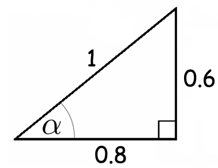
c) Is it a coincidence that in this triangle with hypotenuse 1, sides appear to have the same values as $\sin\alpha$ and $\cos\alpha$? Would this be true for any right triangle?

Solution: a) $\sin\alpha = \dfrac{6}{10} = 0.6$ and $\cos\alpha = \dfrac{8}{10} = 0.8$



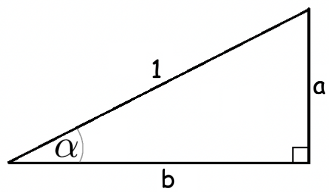
b) To move to a similar triangle, all we are allowed to do is to multiply or divide 6, 8, and 10 by the same number. If we want a hypotenuse 1 unit long, we must divide all three sides by 10. Then we get the triangle with sides 0.6, 0.8, and 1 units long.

$\sin\alpha = \dfrac{0.6}{1} = 0.6$ and $\cos\alpha = \dfrac{0.8}{1} = 0.8$



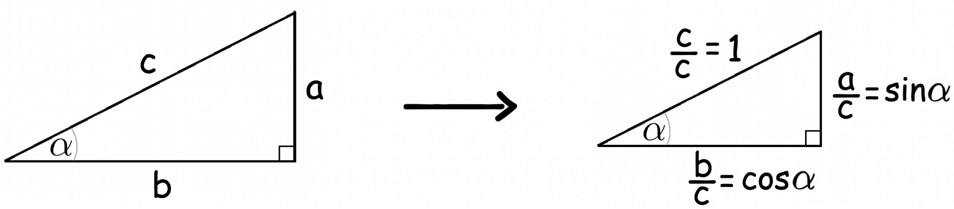
The trigonometric ratios naturally remain the same, because ratios between sides are preserved under similarity. This is exactly the reason why sine, cosine, and the rest are well-defined.

c) This is no coincidence. The easiest division is division by 1. If the hypotenuse of a right triangle is 1, then the trigonometric ratios sine and cosine are the same as the length of the other two sides.



$$\sin\alpha = \frac{a}{1} = a \quad \text{and} \quad \cos\alpha = \frac{b}{1} = b$$

Suppose we have a right triangle with sides a, b, and c, the last one being the hypotenuse. We can create a similar triangle with hypotenuse 1 by dividing all three sides by c. Therefore, for every right triangle, there is a similar one with hypotenuse 1. We will many times prefer right triangles with hypotenuse 1 because then trigonometric ratios can become side lengths.

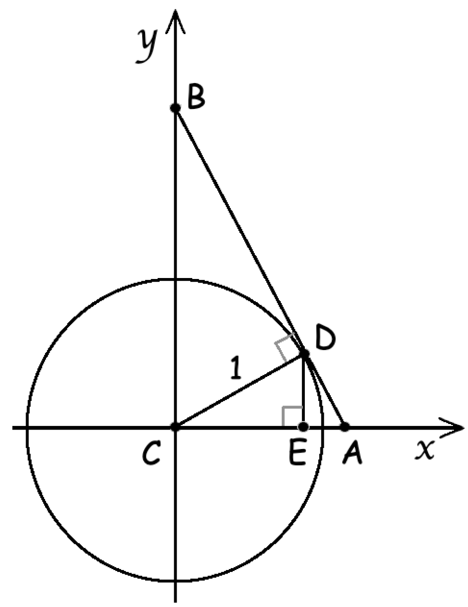


Discussion:

1. Find the exact values of all six trigonometric function values of the angles $30°$ and $60°$.
2. Find approximate values of all six trigonometric function values of the angles $75°$ and $15°$.
3. Some values are recurring. For example, $\sin 30° = \cos 60°$ and $\sin 75° = \cos 15°$. What values repeat and why? Try to state the most general statement expressing this pattern.

Enrichment

1. Consider the unit circle shown on the picture. Line AB is tangent to the circle at point D. Express each of the six trigonometric function values of α as a length of a line segment.
2. Prove that if α is any acute angle, then
   $\sin\alpha + \cos\alpha > 1$.



Sample Problems

1. A surveyor wishes to determine the height of a building. He measures a distance of 1500 feet along the ground from the building, and there he determines the angle of elevation to the top of the building. This angle is $24°$. How tall is the building? Present your answer as
   a) an exact value b) as an approximation, accurate up to five decimal places.
2. A ladder 9 m long is placed against a building. The angle between the ladder and the ground is $65°$. How high up is the top of the ladder? Present your answer as
   a) an exact value b) as an approximation, accurate up to five decimal places.
3. Compute the angles (up to five decimal places) in a triangle with sides 18 m, 13 m, and 13 m long.
4. There is a 10-mile pathway towards the ocean, perpendicular to the beach. Two runners agree to race on this path. The first runner runs the path with a perfect accuracy. The second runner misses the perpendicular path by $1°$ to the right.
   a) How far are apart are the endpoints of the runners' paths?
   b) How much more did the second runner run than the first runner, due to the inaccuracy of the path?

5. From a hot air balloon that is 4000 ft high, the angles of depression to two houses, in line with the balloon, are $47°$ and $32°$. How far apart are the houses?

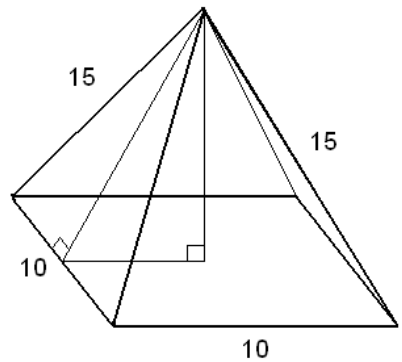
6. Find the area of a regular polygon of 10 sides, inscribed in a circle of radius 5 m.

7. Find the angle that the straight line $y = \frac{2}{3}x + 5$ forms with the positive half of the x-axis.

8. Consider a circle with radius 4 units. The distance between the center of the circle and a point P is 12 units. Compute the angle that is formed by the two tangent lines drawn to the circle from P. Present your answer as

   a) exact value

   b) an approximation, accurate up to four decimal places.



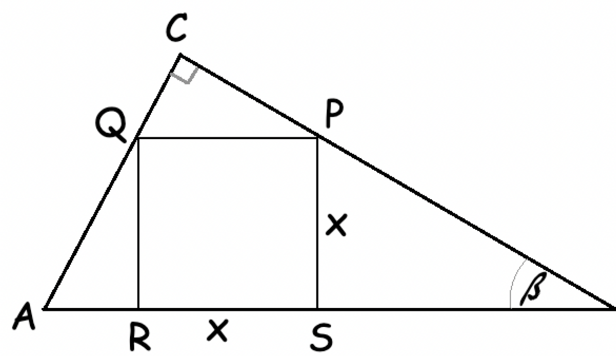
9. The picture shows a straight pyramid with a square base. The sides of the base are 10 in long. The other edges are 15 in long. Find the angle between the base and a triangular face.

10. We are driving toward a tower. The angle of elevation to the top of the tower is $28°$. Then we drive 550 ft toward the tower. Now the angle of elevation is $40°$. How tall is the tower?

11. Suppose that A and B are points at a distance of 10 units from each other. A is the center of a circle with radius 8 units long and B is the center of a circle with radius 4 units long. Find an approximate value of the smaller angle formed by the common tangent lines drawn to the circles.

12. The picture shows a square written into a right triangle. Express each of the given line segments using only x and β.

    AQ, AR, QC, CP, BP, PS



Practice Problems

1. A surveyor wishes to determine the height of a building. He measures a distance of 80 meter along the ground from the building, and there he determines the angle of elevation to the top of the building. This angle is $18°$. How tall is the building? Present your answer as

   a) an exact value b) as an approximation, accurate up to five decimal places.

2. A ladder 5 m long is placed against a building. The angle between the ladder and the building is $15°$. How high up is the top of the ladder? Present your answer as

   a) an exact value b) as an approximation, accurate up to five decimal places.

3. Compute the angles (up to five decimal places) in a triangle with sides 10 m, 21 m, and 21 m long.

4. There is a 26-mile pathway towards the ocean, perpendicular to the beach. Two runners agree to race on this path. The first runner runs the path with a perfect accuracy. The second runner misses the perpendicular path by $1.5°$ to the right.
   a) How far are apart are the endpoints of the runners' paths?
   b) How much more did the second runner run than the first runner, due to the inaccuracy of the path?

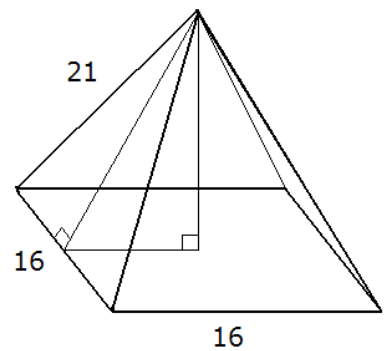
5. From a hot air balloon that is 500m high, the angles of depression to two houses, in line with the balloon, are $38°$ and $49°$. How far apart are the houses?

6. A regular polygon of 12 sides is written into a circle of radius 5ft. Find the area of the polygon.

7. Find the angle that is formed by the positive part of the x-axis and the straight line $2x - 5y = 1$.

8. A point P is 10cm away from the center C of a circle with radius 7cm. Compute the angle that is formed by the two tangent lines drawn to the circle from P. Present your answer as
   a) exact value b) an approximation, accurate up to four decimal places.

9. The picture shows a straight pyramid with a square base. The sides of the base are 16in long. The other edges are 21in long. Find the angle between the base and a triangular face.



10. The angle of elevation to the top of a building changes from $12°$ to $25°$ as the observer travels 200m towards the building. How tall is the building?

Answers

Sample Problems

1. a) $1500 \tan 24° \text{ft}$ b) $667.843\,03\text{ft}$ 2. a) $9 \sin 65° \text{m}$ b) $8.156\,77\text{m}$

3. $\alpha = \beta \approx 46.18694°$ and $\gamma \approx 87.626\,12°$

4. a) 0.17455 mile apart b) the second runner ran 0.00152 mile more than the first runner

5. $10131.398\,461\text{ft}$ 6. $73.473\,16\text{m}^2$ 7. $33.690°$ 8. a) $2 \sin^{-1}\left(\frac{1}{3}\right)$ b) $38.942\,4°$

9. $69.295\,2°$ 10. 798.290ft 11. $47.1564°$

Practice Problems

1. a) $80 \tan 18° \text{m}$ b) $25.993\,58\text{m}$ 2. a) $5 \cos 15° \text{m}$ b) $4.829\,63\text{m}$

3. $76.225\,853°$ $76.225\,853°$ and $27.548\,294°$ 4. a) $0.680\,834$ mile b) $0.008\,9$ mile more

5. $1074.614\,19\text{m}$ 6. 75ft^2 7. $21.801°$ 8. a) $2 \sin^{-1}\left(\frac{7}{10}\right)$ b) $88.854\,0°$

9. $65.668\,15°$ 10. $78.121\,2\text{m}$

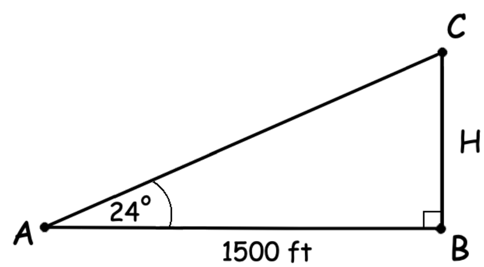
Sample Problems Solutions

1. A surveyor wishes to determine the height of a building. He measures a distance of 1500 feet along the ground from the building, and there he determines the angle of elevation to the top of the building. This angle is $24°$. How tall is the building?

Solution: Consider the right triangle shown on the picture. We denote the height of the building by H.

There is one trigonometric statement connecting $24°$, 1500 feet, and H, and that is $\tan 24°$ stated on the right triangle ABC.

$$\begin{aligned} \tan 24° &= \frac{H}{1500\text{ft}} && \text{multiply by } 1500\text{ft} \\ (1500\text{ft})\tan 24° &= H \\ H &= 1500\tan 24°\text{ft} && \text{exact value} \end{aligned}$$



To compute an approximate value, we enter $1500 \cdot \tan 24$ into our calculator. We need to be careful to enter exactly what we mean (What is the danger in entering $\tan 24 \cdot 1500$?) Also, make sure that your calculator is in degree mode and not in radian or gradient mode. The calculator shows

$$1500\tan 24° \approx 667.843\,027\,963$$

which we present as $667.843\,03$ ft after correctly rounding.

2. A ladder 9m long is placed against a building. The angle between the ladder and the ground is $65°$. How high up is the top of the ladder? Present your answer as

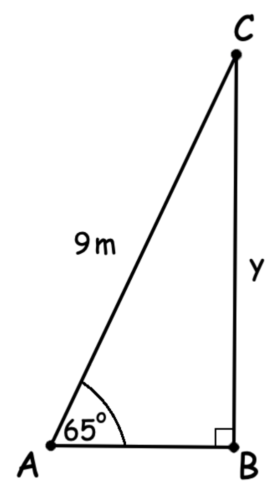
a) an exact value

b) as an approximation, accurate up to five decimal places.

Solution: Consider the right triangle shown on the picture. We denote the height of the top of the ladder by y.
There is one trigonometric statement connecting $65°$, 9 meter, and y, and that is $\sin 65°$ stated on the right triangle ABC.

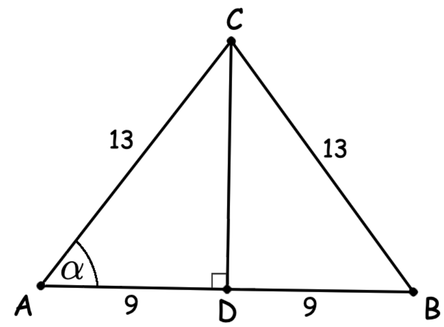
$$\begin{aligned} \sin 65° &= \frac{y}{9\text{m}} && \text{multiply by } 9\text{m} \\ 9\sin 65°\text{m} &= y && \text{this is the exact value} \end{aligned}$$

We enter this into the calculator and obtain the approximation $8.156\,77$m



3. Compute the angles (up to five decimal places) in a triangle with sides 18m, 13m, and 13m long.

   Solution: Consider the picture shown where $AC = BC = 13\text{m}$. Since this triangle isosceles, line CD is a perpendicular bisector of line segment AB. Thus we have a right triangle ADC, with sides $AC = 13\text{m}$ and $AD = 9\text{m}$.

   We can now compute α

$$\cos\alpha = \frac{9}{13} \quad\Longrightarrow\quad \alpha = \cos^{-1}\left(\frac{9}{13}\right) \approx 46.18694^\circ$$

   (Notice that the notations $\cos^{-1}$ and $\arccos$ represent the same thing. You will frequently see both notations.) Since the triangle is isosceles, $\beta = \alpha \approx 46.18694^\circ$. Now

$$\begin{aligned}\alpha + \beta + \gamma &= 180^\circ \\ \gamma &= 180^\circ - (\alpha + \beta) \approx 180^\circ - 2\,(46.18694^\circ) = 87.626\,12^\circ\end{aligned}$$

   Thus $\alpha = \beta \approx 46.18694^\circ$ and $\gamma \approx 87.626\,12^\circ$.

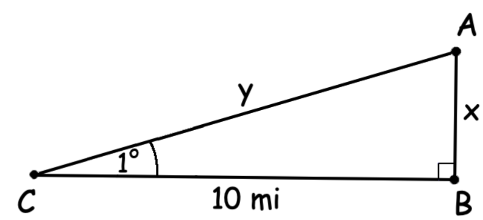
4. There is a 10-mile pathway towards the ocean, perpendicular to the beach. Two runners agree to race on this path. The first runner runs the path with a perfect accuracy. The second runner misses the perpendicular path by 1° to the right.

   a) How far are apart are the endpoints of the runners' paths?

   b) How much more did the second runner run than the first runner, due to the inaccuracy of the path?

   Solution: Consider the right triangle below. Side BC represent the correct path, while side AC is the longer path. Side AB is the distance between the endpoints of the two paths.

$$\begin{aligned}\tan 1^\circ &= \frac{x}{10\text{mi}} \quad\Longrightarrow\quad x = (10\text{mi})\tan 1^\circ \approx 0.17455\text{mi} \\ \cos 1^\circ &= \frac{10\text{mi}}{y} \quad\Longrightarrow\quad y = \frac{10\text{mi}}{\cos 1^\circ} \approx 10.00152\text{mi}\end{aligned}$$

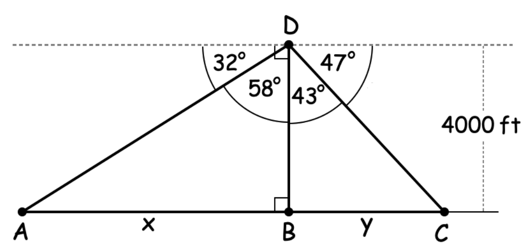
   

   a) the endpoints of the paths are 0.017455 mile apart.

   b) the second runner ran 0.00152 mile more than the first runner

5. From a hot air balloon that is 4000 ft high, the angles of depression to two houses, in line with the balloon, are 47° and 32°. How far apart are the houses?

   Solution: Consider the picture shown. Let D denote the balloon, and A and C denote the two houses. Point B is the point between the houses above which the balloon is located. We first compute the angles inside the triangles as shown on the picture. If the angle of elevation is 32°, then the other angle is its complement, $90^\circ - 32^\circ = 58^\circ$.

   We similarly obtain 58°. We can now compute x and y using the right triangles ABD and BCD.

$$\tan 58^\circ = \frac{x}{4000\text{ft}} \text{ from triangle } ABD \text{ and } \tan 43^\circ = \frac{y}{4000\text{ft}} \text{ from triangle } BCD$$

   The distance between the houses is $x + y$. We solve these equations for x and y and then add.

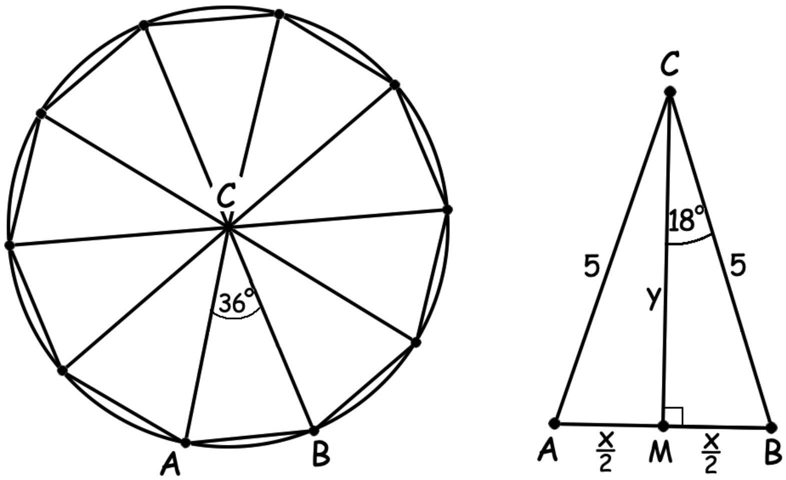
$$\begin{aligned}\tan 58^\circ &= \frac{x}{4000\text{ft}} \quad\Longrightarrow\quad x = 4000\tan 58^\circ\text{ft} \\ \tan 43^\circ &= \frac{y}{4000\text{ft}} \quad\Longrightarrow\quad y = 4000\tan 43^\circ\text{ft}\end{aligned}$$

$$x + y = 4000\tan 58^\circ\text{ft} + 4000\tan 43^\circ\text{ft} = 4000\left(\tan 58^\circ + \tan 43^\circ\right)\text{ft} \approx 10131.398\,461\text{ft}$$

   Caution! $\tan 58^\circ + \tan 43^\circ$ is not the same as $\tan\left(58^\circ + 43^\circ\right)$. It is important to enter the expressions correctly.

6. Find the area of a regular polygon of 10 sides, inscribed in a circle of radius 5 m.

   Solution: We draw a regular polygon of ten sides into a circle. We then connect each vertex with the center of the circle. We will look at one of the ten identical triangles that make up the polygon.



   Consider the isosceles triangle ABC with an angle of $\dfrac{360°}{10} = 36°$ at the center. This triangle does not have a right angle. So we split the triangle in two identical parts by the height drawn to the base. We will find the area of one triangle and then multiply that area by 10 since 10 such triangles make up the polygon. Let us label the base AB by x, and the height CM by y. Then CBM triangle is a right triangle, with $MCB\angle = 18°$, and the hypotenuse is 5 m.

$$\sin 18° = \frac{\frac{x}{2}}{5\text{m}} = \frac{x}{10\text{m}} \quad\Longrightarrow\quad x = (10\text{m})\sin 18°$$

   To find y:

$$\cos 18° = \frac{y}{5\text{m}} \quad\Longrightarrow\quad y = (5\text{m})\cos 18°$$

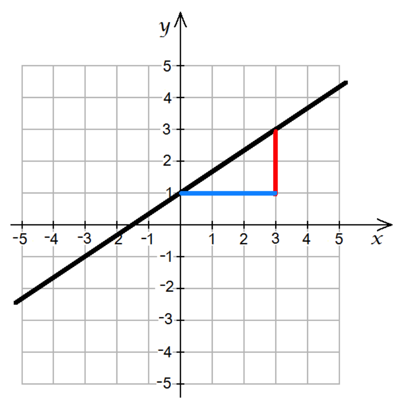
   Thus the triangle AOB has area

$$A = \frac{1}{2}xy = \frac{1}{2}\left(10\sin 18°\text{ m}\right)\left(5\cos 18°\text{ m}\right) = 25\sin 18°\cos 18°\text{ m}^2$$

   Thus the polygon's area is

$$10A = 10\left(25\sin 18°\cos 18°\text{ m}^2\right) = 250\sin 18°\cos 18°\text{ m}^2 \approx 73.473\,16\text{ m}^2$$

7. Find the angle that the straight line $y = \dfrac{2}{3}x + 1$ forms with the positive half of the x-axis.



   Solution: Let α denote the angle between the line and the positive part of the $x-$axis. Consider that the slope means rise over run, i.e.

$$m = \frac{\text{rise}}{\text{run}} = \frac{y_2 - y_1}{x_2 - x_1} = \frac{2}{3}$$

   In the context of trigonometry, the slope will become $\tan\alpha$. Thus we need to enter into the calculator

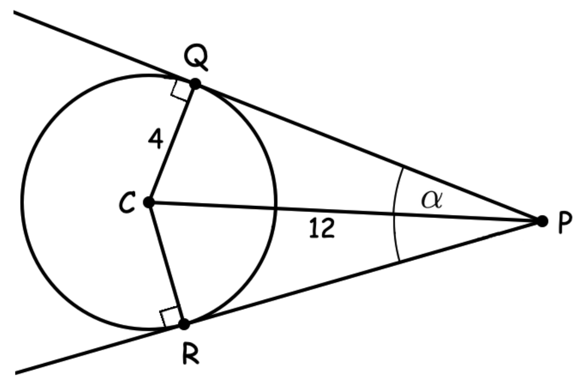
$$\tan^{-1}(2 \div 3) =$$

   and be careful that the calculator is set to degrees. The answer is $33.690°$.

8. Consider a circle with radius 4 units. The distance between the center of the circle and a point P is 12 units. Compute the angle that is formed by the two tangent lines drawn to the circle from P. Present your answer as

   a) exact value b) an approximation, accurate up to four decimal places.

   Solution: Let Q and R denote the points of tangency. Let us also connect the center of the circle C with P.

   We need to notice two things. First, the picture is completely symmetrical to the line CP. Thus the angles QPC and CPR have the same measure. They are both denoted by α, and the angle that we have to compute is simply 2α. We will compute α using the triangle PQC. The second thing we need to notice is that this triangle is a right triangle because the tangent line of a circle is always perpendicular to the radius drawn to the point of tangency. Thus angle $PQC = 90°$. We now easily compute α using right triangle trigonometry:

$$\sin\alpha = \frac{4}{12} = \frac{1}{3} \implies \alpha = \sin^{-1}\left(\frac{1}{3}\right)$$

   The angle we must compute is twice α and so the answer is, as an exact value, $2\alpha = 2\sin^{-1}\left(\frac{1}{3}\right)$ or, using the other type of notation, $2\alpha = 2\arcsin\left(\frac{1}{3}\right)$. To compute the approximate value, we enter this into the calculator and round to the requested accuracy. Make sure the calculator is set in degree mode.

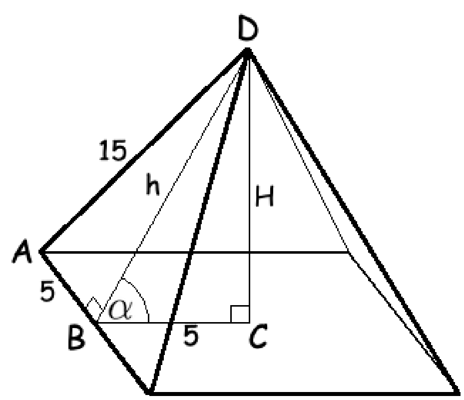
$$2\alpha = (2)\arcsin\left(\frac{1}{3}\right) \approx 38.942\,4°$$

9. The picture shows a straight pyramid with a square base. The sides of the base are 10 in long. The other sides are 15 in long. Find the angle between the base and a triangular face.

   Solution: Let us first label the points and sides we will use as shown below. We need to find angle $\angle DBC = \alpha$.

   We compute the height h of a face by the Pythagorean theorem stated for triangle ABD.

$$5^2 + h^2 = 15^2 \implies h = \sqrt{200} = 10\sqrt{2}$$

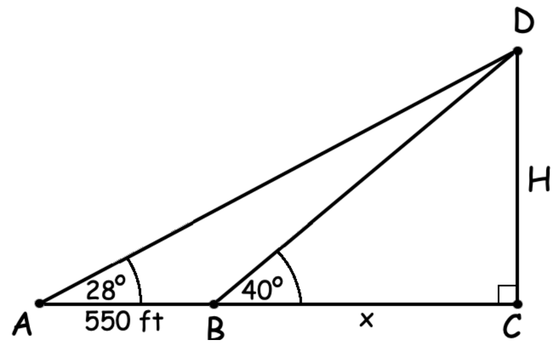
   Now

$$\cos\alpha = \frac{\overline{BC}}{\overline{BD}} = \frac{5}{h} = \frac{5}{10\sqrt{2}} = \frac{1}{2\sqrt{2}}$$

   and so $\alpha = \cos^{-1}\left(\frac{1}{2\sqrt{2}}\right) \approx 69.295\,19°$

10. We are driving toward a tower. The angle of elevation is $28°$. Then we drive 550 ft toward the tower. Now the angle of elevation is $40°$. How tall is the tower?

Solution: Based on the picture below, we can write a system of linear equations.



$$\text{in triangle } ACD \qquad \tan 28° = \frac{H}{x+550}$$

$$\text{in triangle } BCD \qquad \tan 40° = \frac{H}{x}$$

In this system, there are two unknowns, H and x. We will use substitution to solve the system.

It is a bit unusual, but common in trigonometry to symbolically carry the trigonometric expressions and only to substitute approximate values into them in the last line.

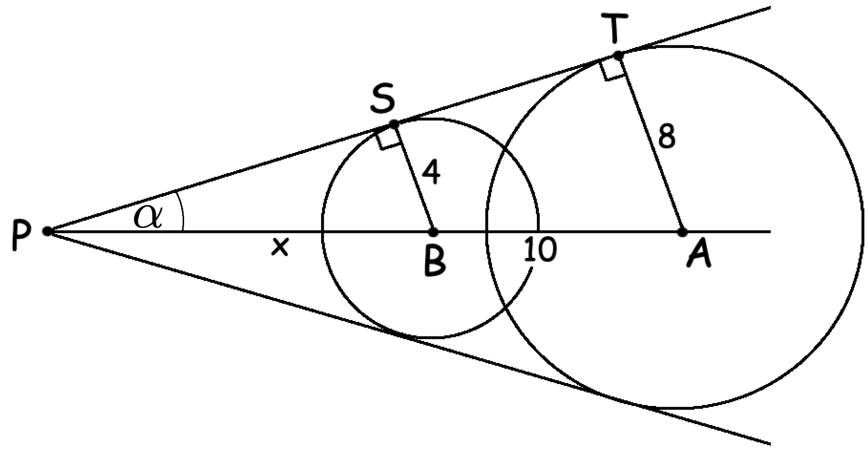
$$\begin{aligned} \tan 28° &= \frac{H}{x+550} \quad \Longrightarrow \quad H = \tan 28° (x+550) \\ \tan 40° &= \frac{H}{x} \quad \Longrightarrow \quad H = \tan 40° x \end{aligned}$$

$$\begin{aligned} \tan 28° (x+550) &= \tan 40° x \\ x \tan 28° + 550 \tan 28° &= x \tan 40° \\ 550 \tan 28° &= x \tan 40° - x \tan 28° \\ 550 \tan 28° &= x (\tan 40° - \tan 28°) \\ \frac{550 \tan 28°}{\tan 40° - \tan 28°} &= x \end{aligned}$$

$$H = \tan 40° x = \tan 40° \left(\frac{550 \tan 28°}{\tan 40° - \tan 28°} \right) = \frac{550 \tan 40° \tan 28°}{\tan 40° - \tan 28°} \approx 798.289\,78 \ \text{(ft)}$$

11. Suppose that A and B are points at a distance of 10 units from each other. A is the center of a circle with radius 8 units long and B is the center of a circle with radius 4 units long. Find an approximate value of the smaller angle formed by the common tangent lines drawn to the circles.

Solution: Let us denote the intersection of the tangent lines by P and the points of tangency by S and T. P is also on the line connection the centers of the circle. Let us denote the distance between B and P by x as shown on the picture.



Triangles PBS and PAT are similar since they both have a right angle and share the angle at P. Therefore, we can find x using similarity:

$$\frac{x+10}{x} = \frac{8}{4}$$

$$x + 10 = 2x$$

$$x = 10$$

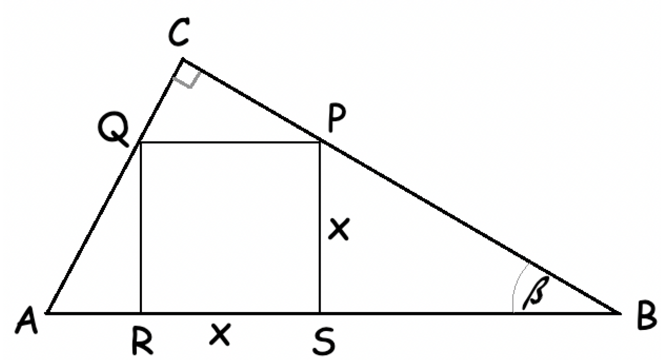
We can now use trigonometry to find angle SPB, denoted by α.
In triangle PBS,

$$\sin\alpha = \frac{4}{10} \quad \text{therefore} \quad \alpha = \sin^{-1}\left(\frac{2}{5}\right) \approx 23.578°$$

The line connecting the centers of the circles is a symmetry line. The angle formed by the two tangent lines is just twice α, that is $2\sin^{-1}\left(\frac{2}{5}\right) \approx 2 \cdot 23.578° = 47.1564°$

12. The picture shows a square written into a right triangle. Express each of the given line segments using only x and β.

AQ, AR, QC, CP, BP, PS



Solution: We leave to the reader to verify that all four triangles, $\triangle\, ABC$, $\triangle\, AQR$, $\triangle\, QPC$, and $\triangle\, BPS$ are similar, and that the angle β has the same measure as angles $\measuredangle AQR$, and $\measuredangle CPQ$. We will take each of the smaller right triangles, and state basic trigonometric statements that include the side x. Then we solve for the other side and hope for the best.

Consider first triangle $\triangle\ AQR$. In that triangle, x plays the role of the longer leg. So $\tan\beta$ and $\cos\beta$ will involve x.

$\cos\beta = \dfrac{x}{AQ}$ and so $AQ = \dfrac{x}{\cos\beta} = x\sec\beta$ and $\tan\beta = \dfrac{AR}{x}$ and so $AR = x\tan\beta$

The third trigonometric function does not include x so it will probably not be useful for us. We move on to triangle PCQ. In that triangle, x is the hypotenuse. So we will state $\sin\beta$ and $\cos\beta$.

$\sin\beta = \dfrac{QC}{x}$ and so $QC = x\sin\beta$ and $\cos\beta = \dfrac{PC}{x}$ and so $PC = x\cos\beta$

Let's consider triangle $\triangle\ PBS$

$\sin\beta = \dfrac{x}{BP}$ and so $BP = \dfrac{x}{\sin\beta} = x\csc\beta$ and $\tan\beta = \dfrac{x}{SB}$ and so $SB = \dfrac{x}{\tan\beta} = x\cot\beta$

