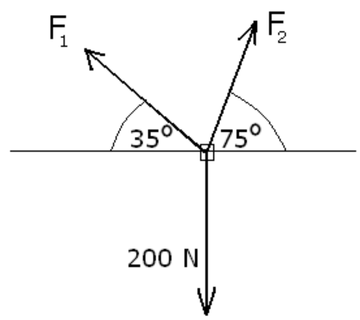
Sample Problems

1. Let $\underline{u} = \underline{i} - 2\underline{j}$ and $\underline{v} = 2\underline{i} + \underline{j}$. Compute each of the following.

   a) $\underline{u} + \underline{v}$ b) $2\underline{u} - \underline{v}$ c) $3\underline{u} - 5\underline{v}$ d) $\underline{u} \cdot \underline{v}$ e) $\underline{u} \cdot (\underline{u} + \underline{v})$

   f) Find M and N so that $M\underline{u} + N\underline{v} = \underline{i}$

2. An object is held by two ropes as shown on the picture below. Find the forces F_1 and F_2 in the rope if the object weighs $200\,\mathrm{N}$, and it is at rest.



3. Compute the angle between the vectors $\underline{a} = 2\underline{i} + 3\underline{j}$ and $\underline{b} = 3\underline{i} - 5\underline{j}$.

4. Prove that if the dot product of two non-zero vectors are zero, then they are perpendicular.

5. Use the dot product to prove that the vectors $\underline{a} = 3\underline{i} - 6\underline{j}$ and $\underline{b} = 10\underline{i} + 5\underline{j}$ are perpendicular.

6. A rhombus is a paralelogram with four equal sides. Prove that the two diagonals in a rhombus are perpendicular.

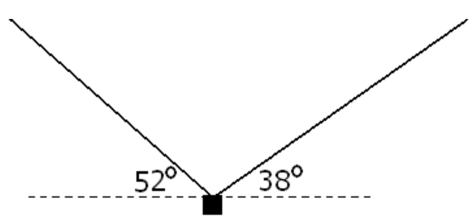
Practice Problems

1. Let $\underline{a} = -3\underline{i} + 2\underline{j}$ and $\underline{b} = 5\underline{i} - \underline{j}$. Compute each of the following.

   a) $\underline{a} - \underline{b}$ b) $3\underline{a} - 2\underline{b}$ c) $|\underline{a}| + |\underline{b}|$ d) $|\underline{a} + \underline{b}|$ e) $\underline{a} \cdot \underline{b}$ f) $(\underline{a} + \underline{b}) \cdot (\underline{b} - \underline{a})$

   g) Find M and N so that $M\underline{a} + N\underline{b} = \underline{i} + \underline{j}$

2. An object is held by two ropes as shown on the picture below. Find the forces F_1 and F_2 in the rope if the object weighs $100\,\mathrm{N}$, and it is at rest.



3. Compute the angle between the vectors $\underline{x} = 2\underline{i} - 5\underline{j}$ and $\underline{b} = \underline{i} + 3\underline{j}$.

Sample Problems - Answers

1.) a) $3\underline{i} - \underline{j}$ b) $-5\underline{j}$ c) $-7\underline{i} - 11\underline{j}$ d) 0 e) 5 f) $M = \frac{1}{5}, N = \frac{2}{5}$

2.) $F_1 \approx 55.085\,90\,\mathrm{N}$ $F_2 \approx 174.344\,680\,\mathrm{N}$ 3.) $115.346\,15°$ 4.) see solutions

5.) see solutions 6.) see solutions

Practice Problems - Answers

1.) a) $-8\underline{i} + 3\underline{j}$ b) $-19\underline{i} + 8\underline{j}$ c) $\sqrt{13} + \sqrt{26}$ d) $\sqrt{5}$ e) -17 f) 13

g) $M = \frac{6}{7}, N = \frac{5}{7}$

2.) $78.801\,1\,\mathrm{N}$ and $61.566\,15\,\mathrm{N}$ 3.) $139.763\,641\,7°$

Sample Problems - Solutions

1. Let $\underline{u} = \underline{i} - 2\underline{j}$ and $\underline{v} = 2\underline{i} + \underline{j}$. Compute each of the following.

   a) $\underline{u} + \underline{v} = \left(\underline{i} - 2\underline{j}\right) + \left(2\underline{i} + \underline{j}\right) = 3\underline{i} - \underline{j}$

   b) $2\underline{u} - \underline{v} = 2\left(\underline{i} - 2\underline{j}\right) - \left(2\underline{i} + \underline{j}\right) = 2\underline{i} + 4\underline{j} - 2\underline{i} - \underline{j} = -5\underline{j}$

   c) $3\underline{u} - 5\underline{v} = 3\left(\underline{i} - 2\underline{j}\right) - 5\left(2\underline{i} + \underline{j}\right) = 3\underline{i} - 6\underline{j} - 10\underline{i} - 5\underline{j} = -7\underline{i} - 11\underline{j}$

   d) $\underline{u} \cdot \underline{v} = \left(\underline{i} - 2\underline{j}\right) \cdot \left(2\underline{i} + \underline{j}\right) = 1 \cdot 2 + (-2) \cdot 1 = 2 - 2 = 0$

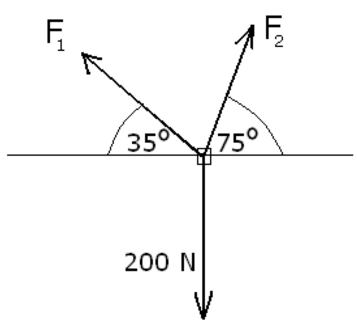
   e) $\underline{u} \cdot (\underline{u} + \underline{v}) = \left(\underline{i} - 2\underline{j}\right) \cdot \left(3\underline{i} - \underline{j}\right) = 1 \cdot 3 + (-2)(-1) = 3 + 2 = 5$

   f) Find M and N so that $M\underline{u} + N\underline{v} = \underline{i}$

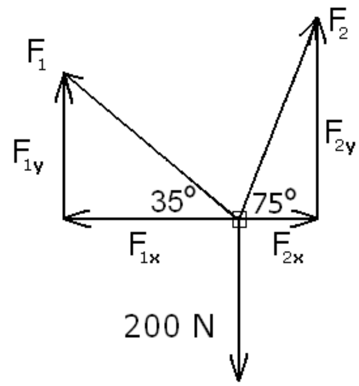
$$\begin{aligned}
M\underline{u} + N\underline{v} &= \underline{i} \\
M\left(\underline{i} - 2\underline{j}\right) + N\left(2\underline{i} + \underline{j}\right) &= \underline{i} \\
M\underline{i} - 2M\underline{j} + 2N\underline{i} + N\underline{j} &= \underline{i} \\
M\underline{i} + 2N\underline{i} - 2M\underline{j} + N\underline{j} &= \underline{i} \\
(M + 2N)\,\underline{i} + (-2M + N)\,\underline{j} &= 1\underline{i} + 0\underline{j} \qquad \Longrightarrow \qquad M + 2N = 1 \quad \text{and} \quad -2M + N = 0
\end{aligned}$$

   This is a linear system of equations that we can solve for M and N, and obtain $M = \dfrac{1}{5}$ and $N = \dfrac{2}{5}$.

2. An object is held by two ropes as shown on the picture below. Find the forces F_1 and F_2 in the rope if the object weighs 200 N, and it is at rest.



   Solution: Let us first decompose forces F_1 and F_2 into horizontal and vertical components. Let us denote the weight of the object by F_3. Recall that upward and right are the positive directions.



Using right triangle trigonometry, we can easily find the length of horizontal and vertical components.

$$\begin{aligned} F_{1x} &= F_1 \cos 35^\circ \qquad & F_{2x} = F_2 \cos 75^\circ \\ F_{1y} &= F_1 \sin 35^\circ \qquad & F_{2y} = F_2 \sin 75^\circ \\ F_3 y &= 200\,\mathrm{N} & \end{aligned}$$

Since the object is at rest, the sum of all forces must be zero F_1. This means that the sum of all horizontal and vertical components are both zero.

$$\begin{aligned} \sum F_x &= 0 \qquad \Longrightarrow \qquad F_1 \cos 35^\circ = F_2 \cos 75^\circ \\ \sum F_y &= 0 \qquad \Longrightarrow \qquad F_1 \sin 35^\circ + F_2 \sin 75^\circ = 200\,\mathrm{N} \end{aligned}$$

Let us express F_2 from the first equation. $F_2 = F_1 \dfrac{\cos 35^\circ}{\cos 75^\circ}$. We substitute this into the second equation.

$$\begin{aligned} F_1 \sin 35^\circ + F_2 \sin 75^\circ &= 200\,\mathrm{N} \\ F_1 \sin 35^\circ + \left(F_1 \frac{\cos 35^\circ}{\cos 75^\circ} \right) \sin 75^\circ &= 200\,\mathrm{N} \\ F_1 \sin 35^\circ + F_1 \frac{\cos 35^\circ \sin 75^\circ}{\cos 75^\circ} &= 200\,\mathrm{N} \\ F_1 \left(\sin 35^\circ + \frac{\cos 35^\circ \sin 75^\circ}{\cos 75^\circ} \right) &= 200\,\mathrm{N} \\ F_1 &= \frac{200\,\mathrm{N}}{\sin 35^\circ + \dfrac{\cos 35^\circ \sin 75^\circ}{\cos 75^\circ}} \approx 55.085\,90\,\mathrm{N} \end{aligned}$$

We can now easily compute F_2.

$$\begin{aligned} F_2 &= F_1 \frac{\cos 35^\circ}{\cos 75^\circ} = \left(\frac{200\,\mathrm{N}}{\sin 35^\circ + \dfrac{\cos 35^\circ \sin 75^\circ}{\cos 75^\circ}} \right) \frac{\cos 35^\circ}{\cos 75^\circ} = \frac{200\,\mathrm{N} \cos 35^\circ}{\sin 35^\circ \cos 75^\circ + \cos 35^\circ \sin 75^\circ} \\ &\approx 174.344\,680\,\mathrm{N} \end{aligned}$$

We may notice that the denominator in F_2 is very symmetrical, in a familiar way. Indeed,

$$F_2 = \frac{200\,\mathrm{N}\cos 35^\circ}{\sin 35^\circ \cos 75^\circ + \cos 35^\circ \sin 75^\circ} = \frac{200\,\mathrm{N}\cos 35^\circ}{\sin(35^\circ + 75^\circ)} = \frac{200\,\mathrm{N}\cos 35^\circ}{\sin 110^\circ}$$

The expression for F_1 can also be simplified by multiplying numerator and denominator by $\cos 75^\circ$.

$$\begin{aligned} F_1 &= \frac{200\,\mathrm{N}}{\sin 35^\circ + \dfrac{\cos 35^\circ \sin 75^\circ}{\cos 75^\circ}} \cdot \frac{\cos 75^\circ}{\cos 75^\circ} = \frac{200\,\mathrm{N}\cos 75^\circ}{\sin 35^\circ \cos 75^\circ + \cos 35^\circ \sin 75^\circ} \\ &= \frac{200\,\mathrm{N}\cos 75^\circ}{\sin(35^\circ + 75^\circ)} = \frac{200\,\mathrm{N}\cos 75^\circ}{\sin 110^\circ} \end{aligned}$$

3. Compute the angle between the vectors $\underline{a} = 2\underline{i} + 3\underline{j}$ and $\underline{b} = 3\underline{i} - 5\underline{j}$.
Solution: Let γ denote the angle between $\underline{a}$ and $\underline{b}$. We will compute the dot product of these vectors using two different methods. On one hand,

$$\underline{a} \cdot \underline{b} = \underline{a} \cdot \underline{b} = 2(3) + 3(-5) = -9$$

On the other hand,

$$\underline{a} \cdot \underline{b} = |\underline{a}|\,|\underline{b}| \cos \gamma \qquad \Longrightarrow \qquad \cos \gamma = \frac{\underline{a} \cdot \underline{b}}{|\underline{a}|\,|\underline{b}|}$$

where $|\underline{a}| = \sqrt{2^2 + 3^2} = \sqrt{13}$ and $|\underline{b}| = \sqrt{3^2 + 5^2} = \sqrt{34}$ and so

$$\cos \gamma = \frac{\underline{a} \cdot \underline{b}}{|\underline{a}|\,|\underline{b}|} = \frac{-9}{\sqrt{13}\sqrt{34}} = -0.428086 \qquad \Longrightarrow \qquad \gamma = \arccos(-0.428086) = 115.346\,15^\circ$$

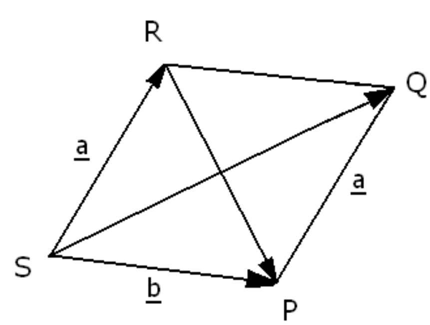
4. Prove that if the dot product of two non-zero vectors are zero, then they are perpendicular.
Solution: Let $\underline{a}$ and $\underline{b}$ denote these vectors and let γ be the smaller angle formed between the angles. Clearly $0^\circ \le \gamma \le 180^\circ$. Since these vectors are not zero, $|\underline{a}| > 0$ and $|\underline{b}| > 0$.

$$\underline{a} \cdot \underline{b} = |\underline{a}|\,|\underline{b}| \cos \gamma = 0$$

Since neither $|\underline{a}|$ nor $|\underline{b}|$ can be zero, we have that $\cos \gamma = 0$. Then $\gamma = 90^\circ + k \cdot 180^\circ$, where k is an integer. The only possible value for $0^\circ \le \gamma \le 180^\circ$ is 90°.

5. Use the dot product to prove that the vectors $\underline{a} = 3\underline{i} - 6\underline{j}$ and $\underline{b} = 10\underline{i} + 5\underline{j}$ are perpendicular.
Solution: $\underline{a} \cdot \underline{b} = \left(3\underline{i} - 6\underline{j}\right) \cdot \left(10\underline{i} + 5\underline{j}\right) = 3 \cdot 10 + (-6)\,5 = 30 - 30 = 0$. Since their dot product is zero, the vectors are perpendicular.

6. A rhombus is a four-sided polygon with four equal sides. Prove that the two diagonals in a rhombus are perpendicular.
Solution: Let P, Q, R, and S denote the vertices of the rhombus as shown below.



Let $\underline{a}$ and $\underline{b}$ denote the sides, with the orientation shown on the picture. Let $x > 0$ denote the length of all sides. Thus $|\underline{a}| = |\underline{b}| = x$ Clearly

$$\overrightarrow{PQ} = \overrightarrow{SR} = \underline{a}, \quad \overrightarrow{SP} = \overrightarrow{RQ} = \underline{b}, \quad \overrightarrow{SQ} = \underline{a} + \underline{b}, \quad \text{and} \quad \overrightarrow{RP} = \underline{b} - \underline{a}$$

The dot product of the two diagonals is

$$\begin{aligned} \overrightarrow{SQ} \cdot \overrightarrow{RP} &= (\underline{a} + \underline{b}) \cdot (\underline{b} - \underline{a}) = \underline{a} \cdot \underline{b} - \underline{a} \cdot \underline{a} + \underline{b} \cdot \underline{b} - \underline{b} \cdot \underline{a} \\ &= -\underline{a} \cdot \underline{a} + \underline{b} \cdot \underline{b} = -|\underline{a}|\,|\underline{a}| + |\underline{b}|\,|\underline{b}| = -x^2 + x^2 = 0 \end{aligned}$$

Since their dot product is zero, the diagonals are perpendicular.