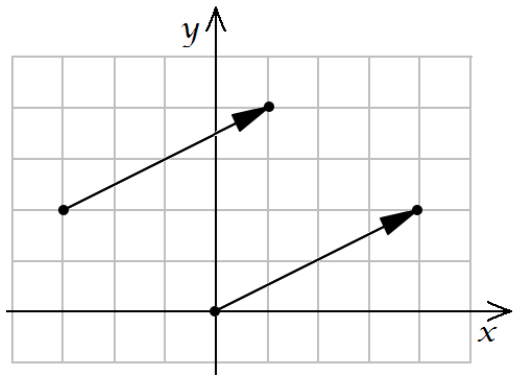
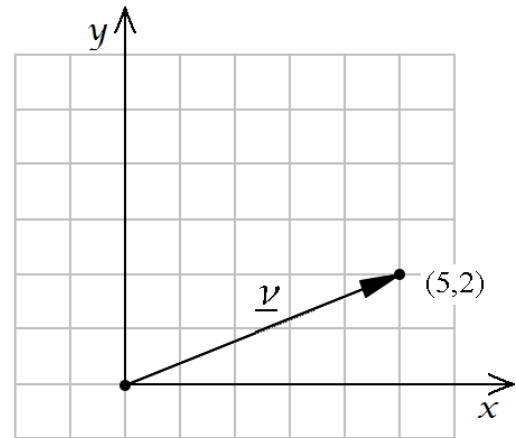


## Vectors Represented Algebraically

**Definition:** A position vector  $\underline{v}$  is represented as  $\underline{v} = \langle a, b \rangle$  is the vector with initial point  $(0, 0)$  and terminal point  $(a, b)$ . The scalars  $a$  and  $b$  are called the components of  $\underline{v}$ .

**Equality of Vectors:** If  $\underline{v} = \langle a, b \rangle$  and  $\underline{w} = \langle c, d \rangle$ , then  $\underline{v} = \underline{w}$  if and only if  $a = c$  and  $b = d$ .



We can write every vector as a position vector. For example, the vector pointing from  $A(-3, 2)$  to  $B(1, 4)$  can be represented as the position vector  $\langle 4, 2 \rangle$ .

The magnitude of a vector  $\underline{v} = \langle a, b \rangle$ , denoted by  $\|\underline{v}\|$ , can be computed as

$$\|\underline{v}\| = \sqrt{a^2 + b^2}$$

If  $\|\underline{v}\| = 1$ , the vector is called a unit vector.

In our example,  $\|\underline{v}\| = \sqrt{5^2 + 2^2} = \sqrt{29}$

Another, very important representation of vectors is by the use of the horizontal and vertical unit vectors. If  $\langle 1, 0 \rangle = \underline{i}$  and  $\langle 0, 1 \rangle = \underline{j}$ , then every vector can be expressed as a linear combination of  $\underline{i}$  and  $\underline{j}$ . For example,  $\langle 5, 2 \rangle = 5\underline{i} + 2\underline{j}$ .

$$\underline{v} = \langle a, b \rangle = a\underline{i} + b\underline{j}$$

$a$  and  $b$  are also called the horizontal and vertical components of  $\underline{v}$ .

## Properties of $\|\underline{v}\|$

If  $\underline{v}$  is a vector and  $\alpha$  is a scalar, then

$$\text{a) } \|\underline{v}\| \geq 0$$

$$\text{c) } \|-\underline{v}\| = \|\underline{v}\|$$

$$\text{b) } \|\underline{v}\| = 0 \text{ if and only if } \underline{v} = \underline{0}.$$

$$\text{d) } \|\alpha\underline{v}\| = |\alpha| \|\underline{v}\|$$

## Operations on Vectors

Suppose that  $\underline{v} = \langle a, b \rangle$  and  $\underline{w} = \langle c, d \rangle$  are vectors and  $\alpha$  is a scalar. The sum and difference of  $\underline{v}$  and  $\underline{w}$  are defined as

$$\underline{v} + \underline{w} = \langle a + c, b + d \rangle = (a + c)\underline{i} + (b + d)\underline{j}$$

$$\underline{v} - \underline{w} = \langle a - c, b - d \rangle = (a - c)\underline{i} + (b - d)\underline{j}$$

$$\alpha\underline{v} = \langle \alpha a, \alpha b \rangle = (\alpha a)\underline{i} + (\alpha b)\underline{j}$$

Example: Suppose that  $\underline{v} = \langle -2, 3 \rangle$  and  $\underline{w} = \langle 1, 4 \rangle$ . Compute each of the following.

$$\text{a) } \underline{v} + \underline{w} \quad \text{b) } \underline{v} - \underline{w} \quad \text{c) } 3\underline{v} - 4\underline{w}$$

$$\text{Solution: a) } \underline{v} + \underline{w} = (-2\underline{i} + 3\underline{j}) + (\underline{i} + 4\underline{j}) = -2\underline{i} + 3\underline{j} + \underline{i} + 4\underline{j} = (-2\underline{i} + \underline{i}) + (3\underline{j} + 4\underline{j}) = -\underline{i} + 7\underline{j}$$

$$\text{b) } \underline{v} - \underline{w} = (-2\underline{i} + 3\underline{j}) - (\underline{i} + 4\underline{j}) = -2\underline{i} + 3\underline{j} - \underline{i} - 4\underline{j} = (-2\underline{i} - \underline{i}) + (3\underline{j} - 4\underline{j}) = -3\underline{i} - \underline{j}$$

$$\text{c) } 3\underline{v} - 4\underline{w} = 3(-2\underline{i} + 3\underline{j}) - 4(\underline{i} + 4\underline{j}) = -6\underline{i} + 9\underline{j} - 4\underline{i} - 16\underline{j} = (-6 - 4)\underline{i} + (9 - 16)\underline{j} = -10\underline{i} - 7\underline{j}$$

## Different Representations of Vectors

Sometimes it is useful to give the magnitude and the direction of a vector. For example, let  $\underline{v}$  be a vector with magnitude 5 units and direction  $120^\circ$  with the positive part of the  $x$ -axis. To express this vector using horizontal and vertical components.

$$\text{horizontal component} = 5 \cos 120^\circ = -2.5$$

$$\text{vertical component} = 5 \sin 120^\circ = 5 \cdot \frac{\sqrt{3}}{2} = \frac{5\sqrt{3}}{2}$$

$$\text{So } \underline{v} = -2.5\underline{i} + \frac{5\sqrt{3}}{2}\underline{j}$$

If a vector is given via horizontal and vertical components, we could also express it by stating the magnitude and the direction of a vector. For example, let  $\underline{w} = 4\underline{i} - \underline{j}$ .

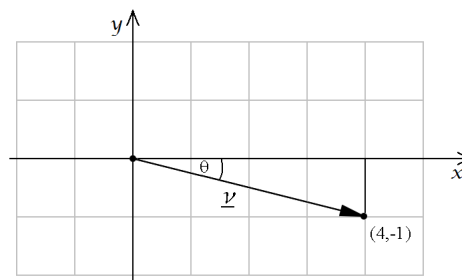
The magnitude is then

$$\|\underline{v}\| = \sqrt{4^2 + (-1)^2} = \sqrt{17}$$

For the direction, we observe that the terminal point is in the fourth quadrant. Then using right triangle trigonometry,

$$\theta = \tan^{-1} \left( \frac{1}{4} \right) \approx 14.036243^\circ$$

and so the direction of the vector is  $-14.036243^\circ$  with respect to the positive part of the  $x$ -axis.



## Finding a Unit Vector

Suppose we need to replace a vector  $\underline{v} = 5\underline{i} - 2\underline{j}$  with a unit vector with the same direction. What we need to do then is divide the vector by its own magnitude. In this case,

$$\frac{\underline{v}}{\|\underline{v}\|} = \frac{5\underline{i} - 2\underline{j}}{\sqrt{(5)^2 + (-2)^2}} = \frac{5\underline{i} - 2\underline{j}}{\sqrt{29}} = \frac{5}{\sqrt{29}}\underline{i} - \frac{2}{\sqrt{29}}\underline{j}$$

This vector is a unit vector since  $\sqrt{\left(\frac{5}{\sqrt{29}}\right)^2 + \left(-\frac{2}{\sqrt{29}}\right)^2} = \sqrt{\frac{25}{29} + \frac{4}{29}} = 1$

## Practice Problems

- Find an algebraic expression for each of the vectors given.
  - the vector pointing from  $A(3, -2)$  to  $B(-2, 9)$ .
  - the vector pointing from  $B(-2, 9)$  to  $A(3, -2)$ .
- Given the vectors  $\underline{u} = 2\underline{i} - \underline{j}$ ,  $\underline{v} = -5\underline{i}$  and  $\underline{w} = \underline{i} + \underline{j}$ , compute each of the following.
 

|                                    |                                                     |                                                       |
|------------------------------------|-----------------------------------------------------|-------------------------------------------------------|
| a) $\underline{u} + \underline{v}$ | c) $2\underline{u} - \underline{v} + \underline{w}$ | e) $5\underline{u} + 3\underline{v} + 5\underline{w}$ |
| b) $\underline{v} - \underline{w}$ | d) $3\underline{w} - 2\underline{v}$                | f) $-2\underline{u} + 3\underline{v} - \underline{w}$ |
- Given the vectors  $\underline{a} = -\underline{i} + 3\underline{j}$ ,  $\underline{b} = \underline{i} - 2\underline{j}$ , and  $\underline{c} = -7\underline{i} + \underline{j}$ , compute each of the following.

- a)  $\underline{a} + \underline{b}$                       d)  $\underline{a} + \underline{b} + \underline{c}$                       g)  $-2\underline{a} + 3\underline{b} - \underline{c}$   
 b)  $3\underline{b}$                       e)  $\underline{a} - 2\underline{b} + \underline{c}$   
 c)  $\underline{b} - \underline{a}$                       f)  $-2\underline{c} = -2\underline{c}$

4. Given the vectors  $\underline{a} = -\underline{i} + 3\underline{j}$ ,  $\underline{b} = \underline{i} - 2\underline{j}$ , and  $\underline{c} = -7\underline{i} + \underline{j}$ , find real numbers  $A$ ,  $B$ , and  $C$ , not all zero, so that

$$A\underline{a} + B\underline{b} + C\underline{c} = \underline{0}$$

5. Find the length and direction for each of the following vectors given.

- a)  $\underline{a} = -\underline{i} + 3\underline{j}$       b)  $\underline{b} = 4\underline{i} + 4\underline{j}$       c)  $\underline{c} = -7\underline{i} - 3\underline{j}$       d)  $\underline{d} = 2\underline{i} - \underline{j}$

6. Express the given vectors using horizontal and vertical components.

- a) magnitude: 10    direction:  $30^\circ$                       c) magnitude: 7    direction:  $270^\circ$   
 b) magnitude: 6    direction:  $135^\circ$                       d) magnitude: 2    direction:  $85^\circ$

7. Find a unit vector with the same direction as the vector given.

- a)  $\underline{v} = -3\underline{i} + 4\underline{j}$       b)  $\underline{w} = -\underline{i} + \underline{j}$       c)  $\underline{u} = 4\underline{i} - 7\underline{j}$

### Answers - Practice Problems

1. a)  $\underline{v} = \langle -5, 11 \rangle = -5\underline{i} + 11\underline{j}$                       b)  $\underline{w} = \langle 5, -11 \rangle = 5\underline{i} - 11\underline{j}$   
 2. a)  $-3\underline{i} - \underline{j}$       b)  $-6\underline{i} - \underline{j}$       c)  $10\underline{i} - \underline{j}$       d)  $13\underline{i} + 3\underline{j}$       e)  $\underline{0}$       f)  $-20\underline{i} + \underline{j}$   
 3. a)  $\underline{j}$       b)  $3\underline{i} - 6\underline{j}$       c)  $2\underline{i} - 5\underline{j}$       d)  $-7\underline{i} + 2\underline{j}$       e)  $-10\underline{i} + 8\underline{j}$       f)  $14\underline{i} - 2\underline{j}$       g)  $12\underline{i} - 13\underline{j}$   
 4. Answers might vary  $A = 13$ ,  $B = 20$ ,  $C = 1$  works.  
 5. a) magnitude:  $\sqrt{10}$     direction:  $\pi - \tan^{-1}(3) \approx 108.434949^\circ$   
 b) magnitude:  $4\sqrt{2}$     direction:  $\frac{\pi}{4} = 45^\circ$   
 c) magnitude:  $\sqrt{58}$     direction:  $\pi + \tan^{-1}\left(\frac{3}{7}\right) \approx 203.1985905^\circ$   
 d) magnitude:  $\sqrt{5}$     direction:  $-\tan^{-1}\left(\frac{1}{2}\right) \approx -26.5650512^\circ$   
 6. a)  $5\underline{i} + 5\sqrt{3}\underline{j}$       b)  $3\sqrt{2}\underline{i} - 3\sqrt{2}\underline{j}$       c)  $-7\underline{j}$       d)  $0.1743115\underline{i} + 1.9924\underline{j}$   
 7. a)  $-\frac{3}{5}\underline{i} + \frac{4}{5}\underline{j}$       b)  $-\frac{\sqrt{2}}{2}\underline{i} + \frac{\sqrt{2}}{2}\underline{j}$       c)  $\frac{4}{\sqrt{65}}\underline{i} - \frac{7}{\sqrt{65}}\underline{j}$