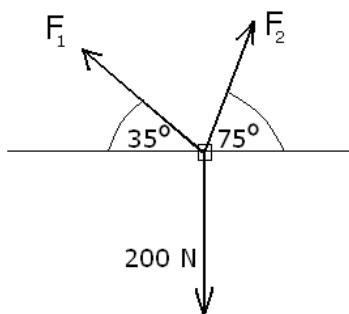


Sample Problems

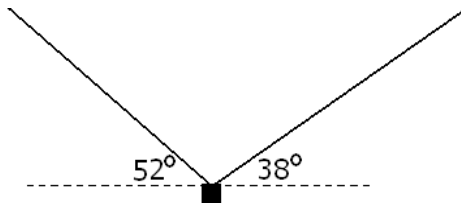
- Let $\underline{u} = \underline{i} - 2\underline{j}$ and $\underline{v} = 2\underline{i} + \underline{j}$. Compute each of the following.
 - $\underline{u} + \underline{v}$
 - $2\underline{u} - \underline{v}$
 - $3\underline{u} - 5\underline{v}$
 - $\underline{u} \cdot \underline{v}$
 - $\underline{u} \cdot (\underline{u} + \underline{v})$
 - Find M and N so that $M\underline{u} + N\underline{v} = \underline{i}$
- An object is held by two ropes as shown on the picture below. Find the forces F_1 and F_2 in the rope if the object weighs 200 N, and it is at rest.



- Compute the angle between the vectors $\underline{a} = 2\underline{i} + 3\underline{j}$ and $\underline{b} = 3\underline{i} - 5\underline{j}$.
- Prove that if the dot product of two non-zero vectors are zero, then they are perpendicular.
- Use the dot product to prove that the vectors $\underline{a} = 3\underline{i} - 6\underline{j}$ and $\underline{b} = 10\underline{i} + 5\underline{j}$ are perpendicular.
- A rhombus is a parallelogram with four equal sides. Prove that the two diagonals in a rhombus are perpendicular.

Practice Problems

- Let $\underline{a} = -3\underline{i} + 2\underline{j}$ and $\underline{b} = 5\underline{i} - \underline{j}$. Compute each of the following.
 - $\underline{a} - \underline{b}$
 - $3\underline{a} - 2\underline{b}$
 - $|\underline{a}| + |\underline{b}|$
 - $|\underline{a} + \underline{b}|$
 - $\underline{a} \cdot \underline{b}$
 - $(\underline{a} + \underline{b}) \cdot (\underline{b} - \underline{a})$
 - Find M and N so that $M\underline{a} + N\underline{b} = \underline{i} + \underline{j}$
- An object is held by two ropes as shown on the picture below. Find the forces F_1 and F_2 in the rope if the object weighs 100 N, and it is at rest.



- Compute the angle between the vectors $\underline{x} = 2\underline{i} - 5\underline{j}$ and $\underline{b} = \underline{i} + 3\underline{j}$.

Sample Problems - Answers

- 1.) a) $3\underline{i} - \underline{j}$ b) $-5\underline{j}$ c) $-7\underline{i} - 11\underline{j}$ d) 0 e) 5 f) $M = \frac{1}{5}, N = \frac{2}{5}$
- 2.) $F_1 \approx 55.08590 \text{ N}$ $F_2 \approx 174.344680 \text{ N}$ 3.) 115.34615° 4.) see solutions
- 5.) see solutions 6.) see solutions

Practice Problems - Answers

- 1.) a) $-8\underline{i} + 3\underline{j}$ b) $-19\underline{i} + 8\underline{j}$ c) $\sqrt{13} + \sqrt{26}$ d) $\sqrt{5}$ e) -17 f) 13 g) $M = \frac{6}{7}, N = \frac{5}{7}$
- 2.) 78.8011 N and 61.56615 N 3.) 139.7636417°

Sample Problems - Solutions

1. Let $\underline{u} = \underline{i} - 2\underline{j}$ and $\underline{v} = 2\underline{i} + \underline{j}$. Compute each of the following.

a) $\underline{u} + \underline{v} = (\underline{i} - 2\underline{j}) + (2\underline{i} + \underline{j}) = 3\underline{i} - \underline{j}$

b) $2\underline{u} - \underline{v} = 2(\underline{i} - 2\underline{j}) - (2\underline{i} + \underline{j}) = 2\underline{i} + 4\underline{j} - 2\underline{i} - \underline{j} = -5\underline{j}$

c) $3\underline{u} - 5\underline{v} = 3(\underline{i} - 2\underline{j}) - 5(2\underline{i} + \underline{j}) = 3\underline{i} - 6\underline{j} - 10\underline{i} - 5\underline{j} = -7\underline{i} - 11\underline{j}$

d) $\underline{u} \cdot \underline{v} = (\underline{i} - 2\underline{j}) \cdot (2\underline{i} + \underline{j}) = 1 \cdot 2 + (-2) \cdot 1 = 2 - 2 = 0$

e) $\underline{u} \cdot (\underline{u} + \underline{v}) = (\underline{i} - 2\underline{j}) \cdot (3\underline{i} - \underline{j}) = 1 \cdot 3 + (-2)(-1) = 3 + 2 = 5$

f) Find M and N so that $M\underline{u} + N\underline{v} = \underline{i}$

$$M\underline{u} + N\underline{v} = \underline{i}$$

$$M(\underline{i} - 2\underline{j}) + N(2\underline{i} + \underline{j}) = \underline{i}$$

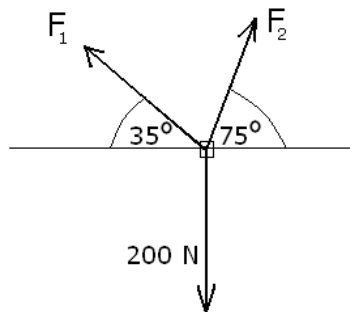
$$M\underline{i} - 2M\underline{j} + 2N\underline{i} + N\underline{j} = \underline{i}$$

$$M\underline{i} + 2N\underline{i} - 2M\underline{j} + N\underline{j} = \underline{i}$$

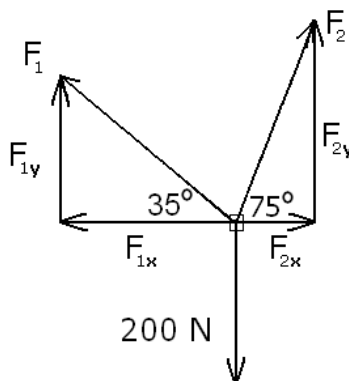
$$(M + 2N)\underline{i} + (-2M + N)\underline{j} = 1\underline{i} + 0\underline{j} \quad \implies \quad M + 2N = 1 \quad \text{and} \quad -2M + N = 0$$

This is a linear system of equations that we can solve for M and N , and obtain $M = \frac{1}{5}$ and $N = \frac{2}{5}$.

2. An object is held by two ropes as shown on the picture below. Find the forces F_1 and F_2 in the rope if the object weighs 200 N, and it is at rest.



Solution: Let us first decompose forces F_1 and F_2 into horizontal and vertical components. Let us denote the weight of the object by F_3 . Recall that upward and right are the positive directions.



Using right triangle trigonometry, we can easily find the length of horizontal and vertical components.

$$\begin{aligned} F_{1x} &= F_1 \cos 35^\circ & F_{2x} &= F_2 \cos 75^\circ \\ F_{1y} &= F_1 \sin 35^\circ & F_{2y} &= F_2 \sin 75^\circ \\ F_{3y} &= 200 \text{ N} \end{aligned}$$

Since the object is at rest, the sum of all forces must be zero F_1 . This means that the sum of all horizontal and vertical components are both zero.

$$\begin{aligned} \sum F_x &= 0 & \implies & F_1 \cos 35^\circ = F_2 \cos 75^\circ \\ \sum F_y &= 0 & \implies & F_1 \sin 35^\circ + F_2 \sin 75^\circ = 200 \text{ N} \end{aligned}$$

Let us express F_2 from the first equation. $F_2 = F_1 \frac{\cos 35^\circ}{\cos 75^\circ}$. We substitute this into the second equation.

$$\begin{aligned} F_1 \sin 35^\circ + F_2 \sin 75^\circ &= 200 \text{ N} \\ F_1 \sin 35^\circ + \left(F_1 \frac{\cos 35^\circ}{\cos 75^\circ} \right) \sin 75^\circ &= 200 \text{ N} \\ F_1 \sin 35^\circ + F_1 \frac{\cos 35^\circ \sin 75^\circ}{\cos 75^\circ} &= 200 \text{ N} \\ F_1 \left(\sin 35^\circ + \frac{\cos 35^\circ \sin 75^\circ}{\cos 75^\circ} \right) &= 200 \text{ N} \\ F_1 &= \frac{200 \text{ N}}{\sin 35^\circ + \frac{\cos 35^\circ \sin 75^\circ}{\cos 75^\circ}} \approx 55.08590 \text{ N} \end{aligned}$$

We can now easily compute F_2 .

$$\begin{aligned} F_2 &= F_1 \frac{\cos 35^\circ}{\cos 75^\circ} = \left(\frac{200 \text{ N}}{\sin 35^\circ + \frac{\cos 35^\circ \sin 75^\circ}{\cos 75^\circ}} \right) \frac{\cos 35^\circ}{\cos 75^\circ} = \frac{200 \text{ N} \cos 35^\circ}{\sin 35^\circ \cos 75^\circ + \cos 35^\circ \sin 75^\circ} \\ &\approx 174.344680 \text{ N} \end{aligned}$$

We may notice that the denominator in F_2 is very symmetrical, in a familiar way. Indeed,

$$F_2 = \frac{200 \text{ N} \cos 35^\circ}{\sin 35^\circ \cos 75^\circ + \cos 35^\circ \sin 75^\circ} = \frac{200 \text{ N} \cos 35^\circ}{\sin (35^\circ + 75^\circ)} = \frac{200 \text{ N} \cos 35^\circ}{\sin 110^\circ}$$

The expression for F_1 can also be simplified by multiplying numerator and denominator by $\cos 75^\circ$.

$$\begin{aligned} F_1 &= \frac{200 \text{ N}}{\sin 35^\circ + \frac{\cos 35^\circ \sin 75^\circ}{\cos 75^\circ}} \cdot \frac{\cos 75^\circ}{\cos 75^\circ} = \frac{200 \text{ N} \cos 75^\circ}{\sin 35^\circ \cos 75^\circ + \cos 35^\circ \sin 75^\circ} \\ &= \frac{200 \text{ N} \cos 75^\circ}{\sin (35^\circ + 75^\circ)} = \frac{200 \text{ N} \cos 75^\circ}{\sin 110^\circ} \end{aligned}$$

3. Compute the angle between the vectors $\underline{a} = 2\underline{i} + 3\underline{j}$ and $\underline{b} = 3\underline{i} - 5\underline{j}$.

Solution: Let γ denote the angle between $\underline{a}$ and $\underline{b}$. We will compute the dot product of these vectors using two different methods. On one hand,

$$\underline{a} \cdot \underline{b} = \underline{a} \cdot \underline{b} = 2(3) + 3(-5) = -9$$

On the other hand,

$$\underline{a} \cdot \underline{b} = |\underline{a}| |\underline{b}| \cos \gamma \implies \cos \gamma = \frac{\underline{a} \cdot \underline{b}}{|\underline{a}| |\underline{b}|}$$

where $|\underline{a}| = \sqrt{2^2 + 3^2} = \sqrt{13}$ and $|\underline{b}| = \sqrt{3^2 + 5^2} = \sqrt{34}$ and so

$$\cos \gamma = \frac{\underline{a} \cdot \underline{b}}{|\underline{a}| |\underline{b}|} = \frac{-9}{\sqrt{13}\sqrt{34}} \approx -0.428086 \implies \gamma = \cos^{-1}\left(\frac{-9}{\sqrt{13}\sqrt{34}}\right) \approx 115.34615^\circ$$

4. Prove that if the dot product of two non-zero vectors are zero, then they are perpendicular.

Solution: Let $\underline{a}$ and $\underline{b}$ denote these vectors and let γ be the smaller angle formed between the angles. Clearly $0^\circ \leq \gamma \leq 180^\circ$. Since these vectors are not zero, $|\underline{a}| > 0$ and $|\underline{b}| > 0$.

$$\underline{a} \cdot \underline{b} = |\underline{a}| |\underline{b}| \cos \gamma = 0$$

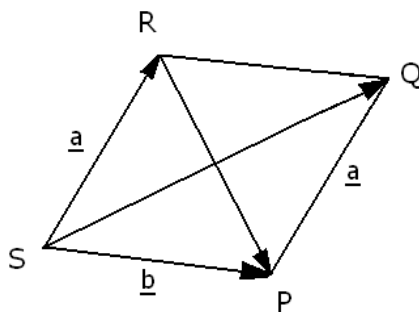
Since neither $|\underline{a}|$ nor $|\underline{b}|$ can be zero, we have that $\cos \gamma = 0$. Then $\gamma = 90^\circ + k \cdot 180^\circ$, where k is an integer. The only possible value for $0^\circ \leq \gamma \leq 180^\circ$ is 90° .

5. Use the dot product to prove that the vectors $\underline{a} = 3\underline{i} - 6\underline{j}$ and $\underline{b} = 10\underline{i} + 5\underline{j}$ are perpendicular.

Solution: $\underline{a} \cdot \underline{b} = (3\underline{i} - 6\underline{j}) \cdot (10\underline{i} + 5\underline{j}) = 3 \cdot 10 + (-6) \cdot 5 = 30 - 30 = 0$. Since their dot product is zero, the vectors are perpendicular.

6. A rhombus is a four-sided polygon with four equal sides. Prove that the two diagonals in a rhombus are perpendicular.

Solution: Let P , Q , R , and S denote the vertices of the rhombus as shown below.



Let $\underline{a}$ and $\underline{b}$ denote the sides, with the orientation shown on the picture. Let $x > 0$ denote the length of all sides. Thus $|\underline{a}| = |\underline{b}| = x$. Clearly

$$\overrightarrow{PQ} = \overrightarrow{SR} = \underline{a}, \quad \overrightarrow{SP} = \overrightarrow{RQ} = \underline{b}, \quad \overrightarrow{SQ} = \underline{a} + \underline{b}, \quad \text{and} \quad \overrightarrow{RP} = \underline{b} - \underline{a}$$

The dot product of the two diagonals is

$$\begin{aligned} \overrightarrow{SQ} \cdot \overrightarrow{RP} &= (\underline{a} + \underline{b}) \cdot (\underline{b} - \underline{a}) = \underline{a} \cdot \underline{b} - \underline{a} \cdot \underline{a} + \underline{b} \cdot \underline{b} - \underline{b} \cdot \underline{a} \\ &= -\underline{a} \cdot \underline{a} + \underline{b} \cdot \underline{b} = -|\underline{a}| |\underline{a}| + |\underline{b}| |\underline{b}| = -x^2 + x^2 = 0 \end{aligned}$$

Since their dot product is zero, the diagonals are perpendicular.