

The following trigonometric identities use the sum, difference, and double-angle formulas for sine, cosine, and tangent.

## Sample Problems

1. (Co-function identities) Prove each of the following identities using the difference formulas for sine and cosine.

a)  $\sin \alpha = \cos \left( \frac{\pi}{2} - \alpha \right)$       b)  $\cos \alpha = \sin \left( \frac{\pi}{2} - \alpha \right)$

2. Prove each of the following identities.

a)  $\frac{2}{\sin 2x} = \tan x + \cot x$

e)  $\left( \sin \frac{x}{2} + \cos \frac{x}{2} \right)^2 = 1 + \sin x$

i)  $1 + \tan \alpha \tan \beta = \frac{\cos(\alpha - \beta)}{\cos \alpha \cos \beta}$

b)  $\cos 2x = \frac{\cot x - \tan x}{\cot x + \tan x}$

f)  $\frac{\tan 2x}{\tan x} = 1 + \frac{1}{\cos 2x}$

j)  $\sin 35^\circ + \sin 25^\circ = \cos 5^\circ$

c)  $\frac{1 + \cos 2x}{1 - \cos 2x} = \cot^2 x$

g)  $\frac{2}{1 - \tan^2 x} = \frac{1 + \cos 2x}{\cos 2x}$

k)  $\cos 12^\circ - \cos 48^\circ = \sin 18^\circ$

d)  $\sin x = \frac{2 \tan \frac{x}{2}}{1 + \tan^2 \frac{x}{2}}$

h)  $\frac{\tan \left( \frac{\pi}{4} - x \right)}{\tan \left( \frac{\pi}{4} + x \right)} = \frac{1 - \sin 2x}{1 + \sin 2x}$

3. Compute the exact value of  $\frac{1 + \tan 15^\circ}{1 - \tan 15^\circ}$ .

4. Compute the exact value of  $\tan \alpha$  if  $\sin(\alpha + 60^\circ) = \cos \alpha$

5. Prove that if  $\sin(x + y) = 3 \sin(x - y)$ , then  $\tan x = 2 \tan y$ .

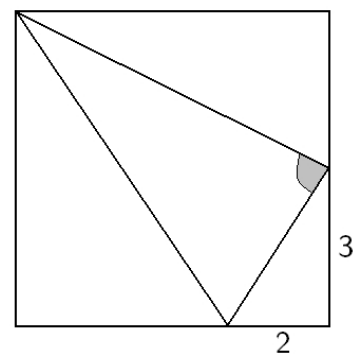
6. Given that  $\frac{\cos(\alpha - \beta)}{\cos(\alpha + \beta)} = \frac{3}{5}$  and  $\tan \alpha = \frac{1}{2}$ , find the exact value of  $\tan \beta$ .

7. Given that  $\tan \alpha$  and  $\tan \beta$  are the solutions of the equation  $x^2 + 6x + 7 = 0$ , show that  $\sin(\alpha + \beta) = \cos(\alpha + \beta)$ .

8. In triangle ABC,  $2 \cos \beta \sin \gamma = \sin \alpha$ . Prove that the triangle is isosceles.

9. Given that  $\alpha + \beta + \gamma = 90^\circ$ , where  $\alpha$ ,  $\beta$ , and  $\gamma$  are acute angles. Prove that  $\cot \alpha \cot \beta \cot \gamma = \cot \alpha + \cot \beta + \cot \gamma$ .

10. Find the exact value of the tangent of the angle shaded on the picture. The quadrilateral is a square with sides 6 units long.



11. Find the exact value of  $\sin 2x$  if we know that  $1 + \tan x = \frac{35}{12} \sin x$ .

## Sample Problems - Solutions

1. (Co-function identities) Prove each of the following identities using the difference formulas for sine and cosine.

a)  $\sin \alpha = \cos \left( \frac{\pi}{2} - \alpha \right)$

Proof:

$$\text{RHS} = \cos \left( \frac{\pi}{2} - \alpha \right) = \cos \left( \frac{\pi}{2} \right) \cos \alpha + \sin \left( \frac{\pi}{2} \right) \sin \alpha = 0 \cdot \cos \alpha + 1 \cdot \sin \alpha = \sin \alpha = \text{LHS}$$

b)  $\cos \alpha = \sin \left( \frac{\pi}{2} - \alpha \right)$

Proof:

$$\text{RHS} = \sin \left( \frac{\pi}{2} - \alpha \right) = \sin \left( \frac{\pi}{2} \right) \cos \alpha - \cos \left( \frac{\pi}{2} \right) \sin \alpha = 1 \cdot \cos \alpha - 0 \cdot \sin \alpha = \cos \alpha = \text{LHS}$$

2. Prove each of the following identities.

a)  $\frac{2}{\sin 2x} = \tan x + \cot x$

Proof:

$$\begin{aligned} \text{LHS} &= \frac{2}{\sin 2x} = \frac{2}{2 \sin x \cos x} = \frac{1}{\sin x \cos x} = \frac{\sin^2 x + \cos^2 x}{\sin x \cos x} = \frac{\sin^2 x}{\sin x \cos x} + \frac{\cos^2 x}{\sin x \cos x} = \frac{\sin x}{\cos x} + \frac{\cos x}{\sin x} \\ &= \tan x + \cot x = \text{RHS} \end{aligned}$$

b)  $\cos 2x = \frac{\cot x - \tan x}{\cot x + \tan x}$

Proof:

$$\begin{aligned} \text{RHS} &= \frac{\cot x - \tan x}{\cot x + \tan x} = \frac{\frac{\cos x}{\sin x} - \frac{\sin x}{\cos x}}{\frac{\cos x}{\sin x} + \frac{\sin x}{\cos x}} = \frac{\frac{\cos^2 x - \sin^2 x}{\sin x \cos x}}{\frac{\cos^2 x + \sin^2 x}{\sin x \cos x}} = \frac{\cos^2 x - \sin^2 x}{\sin x \cos x} \cdot \frac{\sin x \cos x}{\cos^2 x + \sin^2 x} \\ &= \frac{\cos^2 x - \sin^2 x}{\cos^2 x + \sin^2 x} = \frac{\cos 2x}{1} = \text{LHS} \end{aligned}$$

c)  $\frac{1 + \cos 2x}{1 - \cos 2x} = \cot^2 x$

Proof: Recall that the double-angle formula for cosine has three forms:

$$\cos 2x = \cos^2 x - \sin^2 x \quad \text{and} \quad \cos 2x = 2 \cos^2 x - 1 \quad \text{and} \quad \cos 2x = 1 - 2 \sin^2 x$$

We will use different forms for the numerator and denominator.

$$\text{LHS} = \frac{1 + \cos 2x}{1 - \cos 2x} = \frac{1 + 2 \cos^2 x - 1}{1 - (1 - 2 \sin^2 x)} = \frac{2 \cos^2 x}{2 \sin^2 x} = \frac{\cos^2 x}{\sin^2 x} = \cot^2 x = \text{RHS}$$

d)  $\sin x = \frac{2 \tan \frac{x}{2}}{1 + \tan^2 \frac{x}{2}}$

Proof: This identity can look much less intimidating if we introduce a new variable. Let  $\alpha = \frac{x}{2}$ . If we multiply both sides by 2, then  $2\alpha = x$ . Using  $\alpha$  instead of  $x$ , the identity

$$\sin x = \frac{2 \tan \frac{x}{2}}{1 + \tan^2 \frac{x}{2}} \quad \text{becomes} \quad \sin 2\alpha = \frac{2 \tan \alpha}{1 + \tan^2 \alpha}$$

$$\begin{aligned} \text{RHS} &= \frac{2 \tan \frac{x}{2}}{1 + \tan^2 \frac{x}{2}} = \frac{2 \tan \alpha}{1 + \tan^2 \alpha} = \frac{\frac{2 \sin \alpha}{\cos \alpha}}{1 + \frac{\sin^2 \alpha}{\cos^2 \alpha}} = \frac{\frac{2 \sin \alpha}{\cos \alpha}}{\frac{\cos^2 \alpha + \sin^2 \alpha}{\cos^2 \alpha}} = \frac{\frac{2 \sin \alpha}{\cos \alpha}}{\frac{\cos^2 \alpha + \sin^2 \alpha}{\cos^2 \alpha}} = \frac{\frac{2 \sin \alpha}{\cos \alpha}}{\frac{1}{\cos^2 \alpha}} \\ &= \frac{2 \sin \alpha}{\cos \alpha} \cdot \frac{\cos^2 \alpha}{1} = 2 \cos \alpha \sin \alpha = \sin 2\alpha = \sin x = \text{LHS} \end{aligned}$$

$$\text{e) } \left( \sin \frac{x}{2} + \cos \frac{x}{2} \right)^2 = 1 + \sin x$$

Proof: This identity can look much less intimidating if we introduce a new variable. Let  $\alpha = \frac{x}{2}$ . If we multiply both sides by 2, then  $2\alpha = x$ . Using  $\alpha$  instead of  $x$ , the identity

$$\left( \sin \frac{x}{2} + \cos \frac{x}{2} \right)^2 = 1 + \sin x \text{ becomes } (\sin \alpha + \cos \alpha)^2 = 1 + \sin 2\alpha$$

$$\begin{aligned} \text{LHS} &= \left( \sin \frac{x}{2} + \cos \frac{x}{2} \right)^2 = (\sin \alpha + \cos \alpha)^2 = \sin^2 \alpha + \cos^2 \alpha + 2 \sin \alpha \cos \alpha = 1 + 2 \sin \alpha \cos \alpha \\ &= 1 + \sin 2\alpha = 1 + \sin x = \text{RHS} \end{aligned}$$

$$\text{f) } \frac{\tan 2x}{\tan x} = 1 + \frac{1}{\cos 2x}$$

Proof:

$$\begin{aligned} \text{LHS} &= \frac{\tan 2x}{\tan x} = \frac{\frac{2 \tan x}{1 - \tan^2 x}}{\tan x} = \frac{2 \tan x}{1 - \tan^2 x} \cdot \frac{1}{\tan x} = \frac{2}{1 - \tan^2 x} = \frac{2}{1 - \frac{\sin^2 x}{\cos^2 x}} = \frac{2}{\frac{\cos^2 x - \sin^2 x}{\cos^2 x}} \\ &= \frac{2}{\frac{\cos^2 x - \sin^2 x}{\cos^2 x}} = 2 \cdot \frac{\cos^2 x}{\cos^2 x - \sin^2 x} = \frac{2 \cos^2 x}{\cos^2 x - \sin^2 x} = \frac{\cos^2 x + \cos^2 x}{\cos^2 x - \sin^2 x} \\ &= \frac{\cos^2 x + \cos^2 x + \sin^2 x - \sin^2 x}{\cos^2 x - \sin^2 x} = \frac{\cos^2 x - \sin^2 x}{\cos^2 x - \sin^2 x} + \frac{\cos^2 x + \sin^2 x}{\cos^2 x - \sin^2 x} \\ &= 1 + \frac{\cos^2 x + \sin^2 x}{\cos^2 x - \sin^2 x} = 1 + \frac{1}{\cos 2x} = \text{RHS} \end{aligned}$$

$$\text{g) } \frac{2}{1 - \tan^2 x} = \frac{1 + \cos 2x}{\cos 2x}$$

Proof:

$$\text{RHS} = \frac{1 + \cos 2x}{\cos 2x} = \frac{1 + \cos 2x}{\cos^2 x - \sin^2 x} = \frac{1 + 2 \cos^2 x - 1}{\cos^2 x - \sin^2 x} = \frac{2 \cos^2 x}{\cos^2 x - \sin^2 x}$$

Next, we will divide both numerator and denominator by  $\cos^2 x$ .

$$\frac{2 \cos^2 x}{\cos^2 x - \sin^2 x} = \frac{\frac{2 \cos^2 x}{\cos^2 x}}{\frac{\cos^2 x - \sin^2 x}{\cos^2 x}} = \frac{2}{\frac{\cos^2 x}{\cos^2 x} - \frac{\sin^2 x}{\cos^2 x}} = \frac{2}{1 - \tan^2 x} = \text{LHS}$$

$$\text{h) } \frac{\tan\left(\frac{\pi}{4} - x\right)}{\tan\left(\frac{\pi}{4} + x\right)} = \frac{1 - \sin 2x}{1 + \sin 2x}$$

Proof:

$$\begin{aligned} \text{LHS} &= \frac{\tan\left(\frac{\pi}{4} - x\right)}{\tan\left(\frac{\pi}{4} + x\right)} = \frac{\frac{1 - \tan x}{1 + \tan x}}{\frac{\tan x + 1}{1 - \tan x}} = \frac{1 - \tan x}{1 + \tan x} \cdot \frac{1 - \tan x}{1 + \tan x} = \left(\frac{1 - \tan x}{1 + \tan x}\right)^2 = \left(\frac{1 - \frac{\sin x}{\cos x}}{1 + \frac{\sin x}{\cos x}}\right)^2 \\ &= \left(\frac{\frac{\cos x}{\cos x} - \frac{\sin x}{\cos x}}{\frac{\cos x}{\cos x} + \frac{\sin x}{\cos x}}\right)^2 = \left(\frac{\frac{\cos x - \sin x}{\cos x}}{\frac{\cos x + \sin x}{\cos x}}\right)^2 = \left(\frac{\cos x - \sin x}{\cos x + \sin x} \cdot \frac{\cos x}{\cos x}\right)^2 = \left(\frac{\cos x - \sin x}{\cos x + \sin x}\right)^2 \\ &= \frac{1 - \sin 2x}{1 + \sin 2x} = \text{RHS} \end{aligned}$$

$$\text{i) } 1 + \tan \alpha \tan \beta = \frac{\cos(\alpha - \beta)}{\cos \alpha \cos \beta}$$

$$\begin{aligned} \text{LHS} &= 1 + \tan \alpha \tan \beta = 1 + \frac{\sin \alpha \sin \beta}{\cos \alpha \cos \beta} = \frac{\cos \alpha \cos \beta}{\cos \alpha \cos \beta} + \frac{\sin \alpha \sin \beta}{\cos \alpha \cos \beta} = \frac{\cos \alpha \cos \beta + \sin \alpha \sin \beta}{\cos \alpha \cos \beta} \\ &= \frac{\cos(\alpha - \beta)}{\cos \alpha \cos \beta} = \text{RHS} \end{aligned}$$

$$\text{j) } \sin 35^\circ + \sin 25^\circ = \cos 5^\circ$$

$$\begin{aligned} \sin 35^\circ + \sin 25^\circ &= \sin(30^\circ + 5^\circ) + \sin(30^\circ - 5^\circ) \\ &= \sin 30^\circ \cos 5^\circ + \cos 30^\circ \sin 5^\circ + \sin 30^\circ \cos 5^\circ - \cos 30^\circ \sin 5^\circ \\ &= 2 \sin 30^\circ \cos 5^\circ = 2 \left(\frac{1}{2}\right) \cos 5^\circ = \cos 5^\circ \end{aligned}$$

$$\text{k) } \cos 12^\circ - \cos 48^\circ = \sin 18^\circ$$

$$\begin{aligned} \cos 12^\circ - \cos 48^\circ &= \cos(30^\circ - 18^\circ) - \cos(30^\circ + 18^\circ) \\ &= \cos 30^\circ \cos 18^\circ + \sin 30^\circ \sin 18^\circ - (\cos 30^\circ \cos 18^\circ - \sin 30^\circ \sin 18^\circ) \\ &= \cos 30^\circ \cos 18^\circ + \sin 30^\circ \sin 18^\circ - \cos 30^\circ \cos 18^\circ + \sin 30^\circ \sin 18^\circ \\ &= 2 \sin 30^\circ \sin 18^\circ = 2 \left(\frac{1}{2}\right) \sin 18^\circ = \sin 18^\circ \end{aligned}$$

$$3. \text{ Compute the exact value of } \frac{1 + \tan 15^\circ}{1 - \tan 15^\circ}.$$

Solution 1: Let us first compute  $\tan 15^\circ$ .

$$\begin{aligned} \tan 15^\circ &= \tan(45^\circ - 30^\circ) = \frac{\tan 45^\circ - \tan 30^\circ}{1 + \tan 45^\circ \tan 30^\circ} = \frac{1 - \frac{1}{\sqrt{3}}}{1 + \frac{1}{\sqrt{3}}} = \frac{1 - \frac{1}{\sqrt{3}}}{1 + \frac{1}{\sqrt{3}}} \cdot \frac{\sqrt{3}}{\sqrt{3}} \\ &= \frac{\sqrt{3} - 1}{\sqrt{3} + 1} = \frac{\sqrt{3} - 1}{\sqrt{3} + 1} \cdot \frac{\sqrt{3} - 1}{\sqrt{3} - 1} = \frac{3 + 1 - 2\sqrt{3}}{2} = \frac{4 - 2\sqrt{3}}{2} = \frac{2(2 - \sqrt{3})}{2} = 2 - \sqrt{3} \end{aligned}$$

Now our expression becomes

$$\frac{1 + \tan 15^\circ}{1 - \tan 15^\circ} = \frac{1 + (2 - \sqrt{3})}{1 - (2 - \sqrt{3})} = \frac{3 - \sqrt{3}}{-1 + \sqrt{3}} = \frac{\sqrt{3}(\sqrt{3} - 1)}{\sqrt{3} - 1} = \sqrt{3}$$

Solution 2: (a very neat trick) With a little ingenuity, we can turn this expression into a sum formula for tangent.

We re-write 1 as  $\tan 45^\circ$ .

$$\frac{1 + \tan 15^\circ}{1 - \tan 15^\circ} = \frac{1 + \tan 15^\circ}{1 - 1 \cdot \tan 15^\circ} = \frac{\tan 45^\circ + \tan 15^\circ}{1 - \tan 45^\circ \cdot \tan 15^\circ} = \tan(45^\circ + 15^\circ) = \tan 60^\circ = \sqrt{3}$$

Solution 3: Let  $x$  denote  $15^\circ$

$$\begin{aligned} \frac{1 + \tan x}{1 - \tan x} &= \frac{1 + \frac{\sin x}{\cos x}}{1 - \frac{\sin x}{\cos x}} = \frac{1 + \frac{\sin x}{\cos x}}{1 - \frac{\sin x}{\cos x}} \cdot \frac{\cos x}{\cos x} = \frac{\cos x + \sin x}{\cos x - \sin x} \\ &= \frac{\cos x + \sin x}{\cos x - \sin x} \cdot \frac{\cos x + \sin x}{\cos x + \sin x} = \frac{\cos^2 x + \sin^2 x + 2 \sin x \cos x}{\cos^2 x - \sin^2 x} = \frac{1 + \sin 2x}{\cos 2x} \end{aligned}$$

In this case,  $x = 15^\circ$  means that

$$\frac{1 + \tan 15^\circ}{1 - \tan 15^\circ} = \frac{1 + \sin 30^\circ}{\cos 30^\circ} = \frac{1 + \frac{1}{2}}{\frac{\sqrt{3}}{2}} = \frac{\frac{3}{2}}{\frac{\sqrt{3}}{2}} = \frac{3}{2} \cdot \frac{2}{\sqrt{3}} = \frac{3}{\sqrt{3}} = \sqrt{3}$$

4. Compute the exact value of  $\tan \alpha$  if  $\sin(\alpha + 60^\circ) = \cos \alpha$

Solution:

$$\begin{aligned} \sin(\alpha + 60^\circ) &= \cos \alpha \\ \sin \alpha \cos 60^\circ + \cos \alpha \sin 60^\circ &= \cos \alpha \\ \sin \alpha \left(\frac{1}{2}\right) + \cos \alpha \left(\frac{\sqrt{3}}{2}\right) &= \cos \alpha \\ \frac{1}{2} \sin \alpha &= \cos \alpha - \cos \alpha \left(\frac{\sqrt{3}}{2}\right) \\ \frac{1}{2} \sin \alpha &= \left(1 - \frac{\sqrt{3}}{2}\right) \cos \alpha \end{aligned}$$

Case 1. If  $\cos \alpha = 0$ , then clearly

$$\begin{aligned} \frac{1}{2} \sin \alpha &= \left(1 - \frac{\sqrt{3}}{2}\right) \cdot 0 \\ \frac{1}{2} \sin \alpha &= 0 \\ \sin \alpha &= 0 \end{aligned}$$

That's impossible since  $\sin^2 \alpha + \cos^2 \alpha = 1$ .

Case 2. If  $\cos \alpha \neq 0$ , then we can divide by it.

$$\begin{aligned} \frac{1}{2} \sin \alpha &= \left(1 - \frac{\sqrt{3}}{2}\right) \cos \alpha && \text{divide by } \cos \alpha \\ \frac{1}{2} \frac{\sin \alpha}{\cos \alpha} &= \left(1 - \frac{\sqrt{3}}{2}\right) && \text{multiply by 2} \\ \tan \alpha &= 2 \left(1 - \frac{\sqrt{3}}{2}\right) = 2 - \sqrt{3} \end{aligned}$$

5. Prove that if  $\sin(x + y) = 3 \sin(x - y)$ , then  $\tan x = 2 \tan y$ .

Solution:

$$\begin{aligned} \sin(x + y) &= 3 \sin(x - y) \\ \sin x \cos y + \cos x \sin y &= 3 \sin x \cos y - 3 \cos x \sin y && \text{add } 3 \cos x \sin y \\ \sin x \cos y + 4 \cos x \sin y &= 3 \sin x \cos y && \text{subtract } \sin x \cos y \\ 4 \cos x \sin y &= 2 \sin x \cos y && \text{divide by } \cos x \cos y \\ \frac{4 \cos x \sin y}{\cos x \cos y} &= \frac{2 \sin x \cos y}{\cos x \cos y} \\ 4 \tan y &= 2 \tan x \\ 2 \tan y &= \tan x \end{aligned}$$

6. Given that  $\frac{\cos(\alpha - \beta)}{\cos(\alpha + \beta)} = \frac{3}{5}$  and  $\tan \alpha = \frac{1}{2}$ , find the exact value of  $\tan \beta$ .

Solution:

$$\begin{aligned} \frac{\cos(\alpha - \beta)}{\cos(\alpha + \beta)} &= \frac{3}{5} \\ 5 \cos(\alpha - \beta) &= 3 \cos(\alpha + \beta) \\ 5 \cos \alpha \cos \beta + 5 \sin \alpha \sin \beta &= 3 \cos \alpha \cos \beta - 3 \sin \alpha \sin \beta && \text{add } 3 \sin \alpha \sin \beta \\ 5 \cos \alpha \cos \beta + 8 \sin \alpha \sin \beta &= 3 \cos \alpha \cos \beta && \text{subtract } 5 \cos \alpha \cos \beta \\ 8 \sin \alpha \sin \beta &= -2 \cos \alpha \cos \beta && \text{divide by } 2 \cos \alpha \cos \beta \\ \frac{4 \sin \alpha \sin \beta}{\cos \alpha \cos \beta} &= -\frac{\cos \alpha \cos \beta}{\cos \alpha \cos \beta} \\ 4 \tan \alpha \tan \beta &= -1 && \tan \alpha = \frac{1}{2} \\ 4 \left(\frac{1}{2}\right) \tan \beta &= -1 \\ 2 \tan \beta &= -1 \\ \tan \beta &= -\frac{1}{2} \end{aligned}$$

$$\text{check: } \tan \alpha = \frac{1}{2} \quad \sin \alpha = \frac{1}{\sqrt{5}} \quad \cos \alpha = \frac{2}{\sqrt{5}}$$

$$\tan \beta = -\frac{1}{2} \quad \sin \beta = -\frac{1}{\sqrt{5}} \quad \cos \beta = \frac{2}{\sqrt{5}}$$

$$\frac{\cos(\alpha - \beta)}{\cos(\alpha + \beta)} = \frac{\cos \alpha \cos \beta + \sin \alpha \sin \beta}{\cos \alpha \cos \beta - \sin \alpha \sin \beta} = \frac{\frac{2}{\sqrt{5}} \cdot \frac{2}{\sqrt{5}} + \frac{1}{\sqrt{5}} \left(-\frac{1}{\sqrt{5}}\right)}{\frac{2}{\sqrt{5}} \cdot \frac{2}{\sqrt{5}} - \frac{1}{\sqrt{5}} \left(-\frac{1}{\sqrt{5}}\right)} = \frac{\frac{4}{5} - \frac{1}{5}}{\frac{4}{5} + \frac{1}{5}} = \frac{\frac{3}{5}}{\frac{5}{5}} = \frac{3}{5}$$

7. Given that  $\tan \alpha$  and  $\tan \beta$  are the solutions of the equation  $x^2 + 6x + 7 = 0$ , show that  $\sin(\alpha + \beta) = \cos(\alpha + \beta)$ .

Solution: We first solve the quadratic equation.

$$\begin{aligned} x^2 + 6x + 7 &= 0 \\ x^2 + 6x + 9 - 9 + 7 &= 0 \\ (x + 3)^2 - 2 &= 0 \\ (x + 3)^2 - (\sqrt{2})^2 &= 0 \\ (x + 3 + \sqrt{2})(x + 3 - \sqrt{2}) &= 0 \end{aligned}$$

$$x_1 = -3 - \sqrt{2} \quad \text{and} \quad x_2 = -3 + \sqrt{2}$$

So let's say that  $\tan \alpha = -3 - \sqrt{2}$  and  $\tan \beta = -3 + \sqrt{2}$ . then

$$\tan(\alpha + \beta) = \frac{\tan \alpha + \tan \beta}{1 - \tan \alpha \tan \beta} = \frac{-3 - \sqrt{2} + (-3 + \sqrt{2})}{1 - (-3 - \sqrt{2})(-3 + \sqrt{2})} = \frac{-6}{1 - 7} = 1$$

and since

$$\begin{aligned} \tan(\alpha + \beta) &= 1 \\ \frac{\sin(\alpha + \beta)}{\cos(\alpha + \beta)} &= 1 \\ \sin(\alpha + \beta) &= \cos(\alpha + \beta) \end{aligned}$$

8. In triangle ABC,  $2 \cos \beta \sin \gamma = \sin \alpha$ . Prove that the triangle is isosceles.

Solution:  $\alpha + \beta + \gamma = 180^\circ$ . Let us write  $\alpha$  as  $180^\circ - (\beta + \gamma)$

$$\begin{aligned} 2 \cos \beta \sin \gamma &= \sin \alpha \\ 2 \cos \beta \sin \gamma &= \sin(180^\circ - (\beta + \gamma)) \end{aligned}$$

Recall that for all  $x$ ,  $\sin x = \sin(180^\circ - x)$  and so  $\sin(180^\circ - (\beta + \gamma))$  can be simplified as  $\sin(\beta + \gamma)$ .

$$\begin{aligned} 2 \cos \beta \sin \gamma &= \sin(\beta + \gamma) \\ 2 \cos \beta \sin \gamma &= \sin \beta \cos \gamma + \cos \beta \sin \gamma && \text{subtract } 2 \cos \beta \sin \gamma \\ 0 &= \sin \beta \cos \gamma - \cos \beta \sin \gamma \\ 0 &= \sin(\beta - \gamma) \\ \beta - \gamma &= k \cdot 180^\circ \\ \beta &= \gamma + k \cdot 180^\circ \end{aligned}$$

Since  $\beta$  and  $\gamma$  are both between  $0^\circ$  and  $180^\circ$ , this means that  $k = 0$ . Thus  $\beta = \gamma$  and so  $b = c$  and the triangle is isosceles.

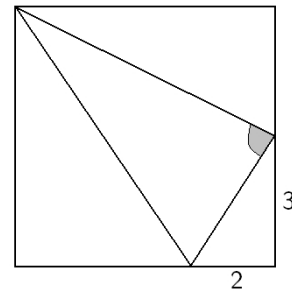
9. Given that  $\alpha + \beta + \gamma = 90^\circ$ , where  $\alpha$ ,  $\beta$ , and  $\gamma$  are acute angles. Prove that  $\cot \alpha \cot \beta \cot \gamma = \cot \alpha + \cot \beta + \cot \gamma$ .

Solution:  $\gamma = 90^\circ - (\beta + \alpha)$  and  $\cot \gamma = \cot(90^\circ - (\beta + \alpha))$ .

$$\cot(90^\circ - (\alpha + \beta)) = \tan(\alpha + \beta) \text{ by the co-function identities}$$

$$\begin{aligned} \text{RHS} &= \cot \alpha + \cot \beta + \cot \gamma = \frac{1}{\tan \alpha} + \frac{1}{\tan \beta} + \tan(\alpha + \beta) \\ &= \frac{1}{\tan \alpha} + \frac{1}{\tan \beta} + \frac{\tan \alpha + \tan \beta}{1 - \tan \alpha \tan \beta} \\ &= \frac{\tan \beta (1 - \tan \alpha \tan \beta)}{\tan \alpha \tan \beta (1 - \tan \alpha \tan \beta)} + \frac{\tan \alpha (1 - \tan \alpha \tan \beta)}{\tan \alpha \tan \beta (1 - \tan \alpha \tan \beta)} + \frac{\tan \alpha \tan \beta (\tan \alpha + \tan \beta)}{\tan \alpha \tan \beta (1 - \tan \alpha \tan \beta)} \\ &= \frac{\tan \beta (1 - \tan \alpha \tan \beta) + \tan \alpha (1 - \tan \alpha \tan \beta) + \tan \alpha \tan \beta (\tan \alpha + \tan \beta)}{\tan \alpha \tan \beta (1 - \tan \alpha \tan \beta)} \\ &= \frac{\tan \beta - \tan \alpha \tan^2 \beta + \tan \alpha - \tan^2 \alpha \tan \beta + \tan^2 \alpha \tan \beta + \tan \alpha \tan^2 \beta}{\tan \alpha \tan \beta (1 - \tan \alpha \tan \beta)} \\ &= \frac{\tan \beta + \tan \alpha}{\tan \alpha \tan \beta (1 - \tan \alpha \tan \beta)} = \frac{1}{\tan \alpha} \cdot \frac{1}{\tan \beta} \cdot \frac{\tan \beta + \tan \alpha}{(1 - \tan \alpha \tan \beta)} = \frac{1}{\tan \alpha} \cdot \frac{1}{\tan \beta} \cdot \tan(\alpha + \beta) \\ &= \frac{1}{\tan \alpha} \cdot \frac{1}{\tan \beta} \cdot \cot(90^\circ - (\alpha + \beta)) = \frac{1}{\tan \alpha} \cdot \frac{1}{\tan \beta} \cdot \cot \gamma = \cot \alpha \cot \beta \cot \gamma = \text{LHS} \end{aligned}$$

10. Find the exact value of the tangent of the angle shaded on the picture. The quadrilateral is a square with sides 6 units long.



Solution: Let us denote our angle by  $\theta$ , and the other angles near it by  $\alpha$  and  $\beta$ , as shown on the picture.

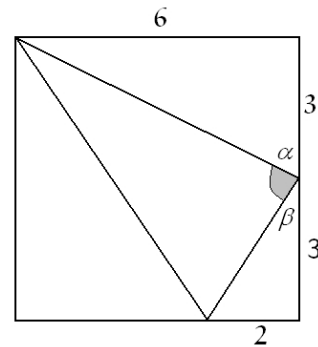
Clearly,  $\theta = 180^\circ - (\alpha + \beta)$ . Then

$$\tan \theta = \tan(180^\circ - (\alpha + \beta)) = -\tan(\alpha + \beta)$$

$$\text{where } \tan \alpha = \frac{6}{3} = 2 \quad \text{and} \quad \tan \beta = \frac{2}{3}.$$

We apply the sum formula for tangent:

$$\begin{aligned} \tan \theta &= -\tan(\alpha + \beta) = -\frac{\tan \alpha + \tan \beta}{1 - \tan \alpha \tan \beta} \\ &= -\frac{2 + \frac{2}{3}}{1 - 2 \cdot \frac{2}{3}} = -\frac{\frac{8}{3}}{-\frac{1}{3}} = 8 \end{aligned}$$



11. Find the exact value of  $\sin 2x$  if we know that  $1 + \tan x = \frac{35}{12} \sin x$ .

Solution:

$$\begin{aligned} 1 + \tan x &= \frac{35}{12} \sin x \\ 1 + \frac{\sin x}{\cos x} &= \frac{35}{12} \sin x && \text{multiply by } \cos x \\ \cos x + \sin x &= \frac{35}{12} \sin x \cos x && \text{square both sides} \\ (\cos x + \sin x)^2 &= \left(\frac{35}{12} \sin x \cos x\right)^2 \\ \cos^2 x + \sin^2 x + 2 \sin x \cos x &= \left(\frac{35}{12}\right)^2 \sin^2 x \cos^2 x \\ 1 + 2 \sin x \cos x &= \left(\frac{35}{12}\right)^2 (\sin x \cos x)^2 && \text{let } a = \sin x \cos x \\ 1 + 2a &= \frac{35^2}{12^2} a^2 && \text{multiply by 144} \\ 144 + 288a &= 1225a^2 \\ 0 &= 1225a^2 - 288a - 144 \end{aligned}$$



We solve the quadratic equation using the quadratic formula:

$$\begin{aligned} a_{1,2} &= \frac{288 \pm \sqrt{(-288)^2 - 4(1225)(-144)}}{2 \cdot 1225} = \frac{288 \pm \sqrt{82\,944 + 705\,600}}{2450} \\ &= \frac{288 \pm \sqrt{788\,544}}{2450} = \frac{288 \pm 888}{2450} = \begin{cases} \frac{288 + 888}{2450} = \frac{1176}{2450} = \frac{12}{25} \\ \frac{288 - 888}{2450} = \frac{-600}{2450} = -\frac{12}{49} \end{cases} \end{aligned}$$

Recall that  $a = \sin x \cos x$ . Then  $\sin 2x = 2 \sin x \cos x = 2a = \frac{24}{25}$  or  $-\frac{24}{49}$ .