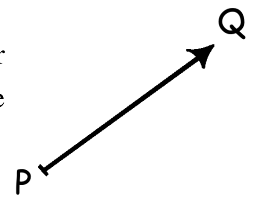
Definition: A vector is a directed line segment.

A vector is determined by its magnitude (or length) and its direction. The vector shown on the picture can be denoted by $\overrightarrow{PQ}$ or by $\vec{v}$ or by $\underline{v}$. The length of the vector $\overrightarrow{PQ}$ is the length of line segment PQ, often denoted by $\overline{PQ}$.



Definition: If a vector has length zero, we call it the zero vector (denoted by $\underline{0}$) and we define its direction to be arbitrary: the zero vector is parallel or perpendicular to any vector.

Definition: Two vectors are equal if they are located on parallel lines, are pointing to the same direction, and have equal magnitudes.

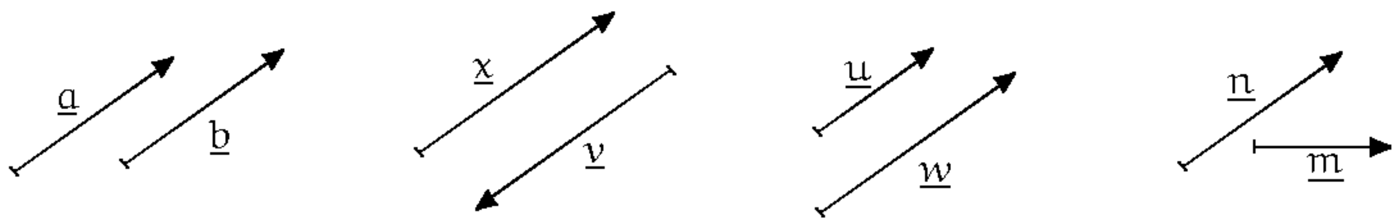
Consider the pairs of vectors shown below.

$\underline{a} = \underline{b}$.

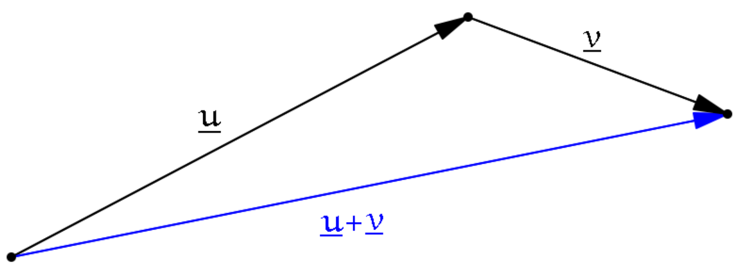
$\underline{x} \neq \underline{v}$ because they are pointing in opposite directions.

$\underline{u} \neq \underline{w}$ because they do not have the same magnitude.

$\underline{n} \neq \underline{m}$ because they are not parallel.



Definition: The sum of two vectors is defined as follows. Let $\underline{u}$ and $\underline{v}$ be two vectors given. We define the sum $\underline{u} + \underline{v}$ by positioning the initial point of $\underline{v}$ at the terminal point of $\underline{u}$. Then $\underline{u} + \underline{v}$ is the vector pointing from the initial point of $\underline{u}$ to the terminal point of $\underline{v}$.

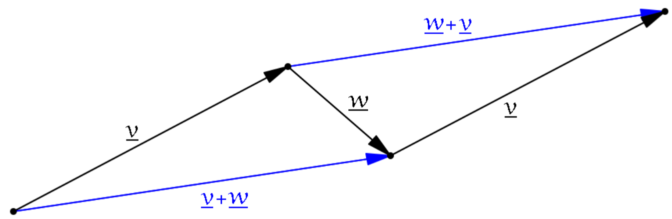


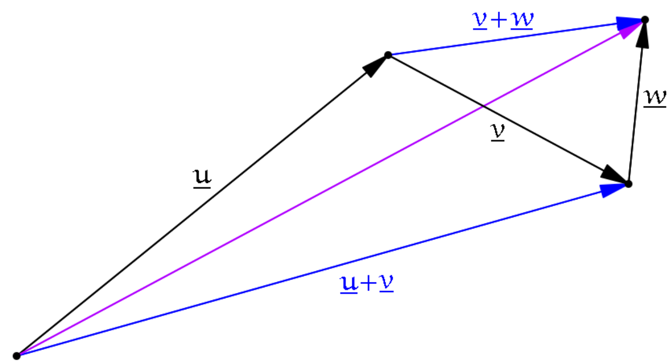
Vector addition has properties similar to addition among real numbers.

For all vectors $\underline{v}$, $\underline{v}+\underline{0}=\underline{v}$ (identity porperty)

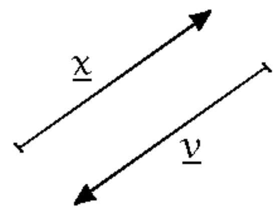
For all vectors $\underline{v}$ and $\underline{w}$, $\underline{v}+\underline{w}=\underline{w}+\underline{v}$ (addition is commutative)

For all vectors $\underline{u}$, $\underline{v}$ and $\underline{w}$, $(\underline{u}+\underline{v})+\underline{w}=\underline{u}+(\underline{v}+\underline{w})$ (addition is associative)





We can define the opposite of a vector as follows: two vectors are opposites if they are parallel, of equal length, and point in the opposite direction. So, from the picture shown, $\underline{x}$ and $\underline{v}$ are opposites. We use the notation familiar from numbers: $\underline{x}=-\underline{v}$.

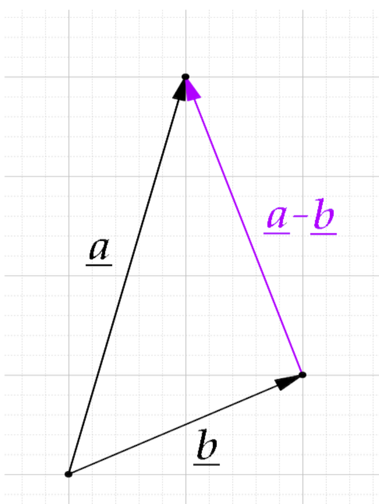


Using this definition, a vector and its opposite add up to the zero vector: $\underline{v}+(-\underline{v})=\underline{0}$.

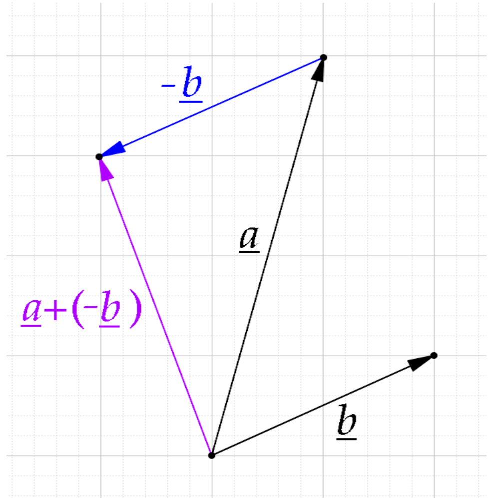
Now that we have the opposite of a vector, we can also define subtraction of vectors. Just as with numbers, to subtract is to add the opposite. There are three different ways to think about subtraction of vectors.

1. The vector $\underline{a}-\underline{b}$ is the vector we obtain by placing the initial points of $\underline{a}$ and $\underline{b}$ at the same place and then drawing the vector from the terminal point of $\underline{b}$ to the terminal point of $\underline{a}$.

2. Let $\underline{x}=\underline{a}-\underline{b}$. If we rearrange the equation, we obtain $\underline{a}=\underline{b}+\underline{x}$. In other words, $\underline{x}$ is the vector that, when added to $\underline{b}$, the result is $\underline{a}$.

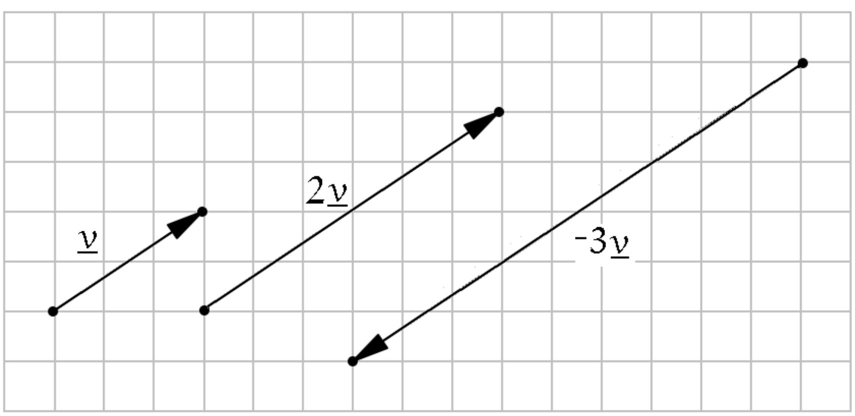


3. To sutract is to add the opposite: $\underline{a} - \underline{b} = \underline{a} + (-\underline{b})$.



Definition: In the context of vectors, we refer to real numbers as scalars. Scalar multiplication is when we multiply a vector by a real number. Suppose that $\underline{v}$ is a vector and c is a real number.

1. If $c > 0$, then $c\underline{v}$ is the vector whose magnitude is c times the magnitude of $\underline{v}$ and whose direction is the same as the direction of $\underline{v}$.
2. If $c < 0$, then $c\underline{v}$ is the vector whose magnitude is c times the magnitude of $\underline{v}$ and whose direction is the opposite of the direction of $\underline{v}$.
3. If $c = 0$, then $c\underline{v} = \underline{0}$, the zero vector.



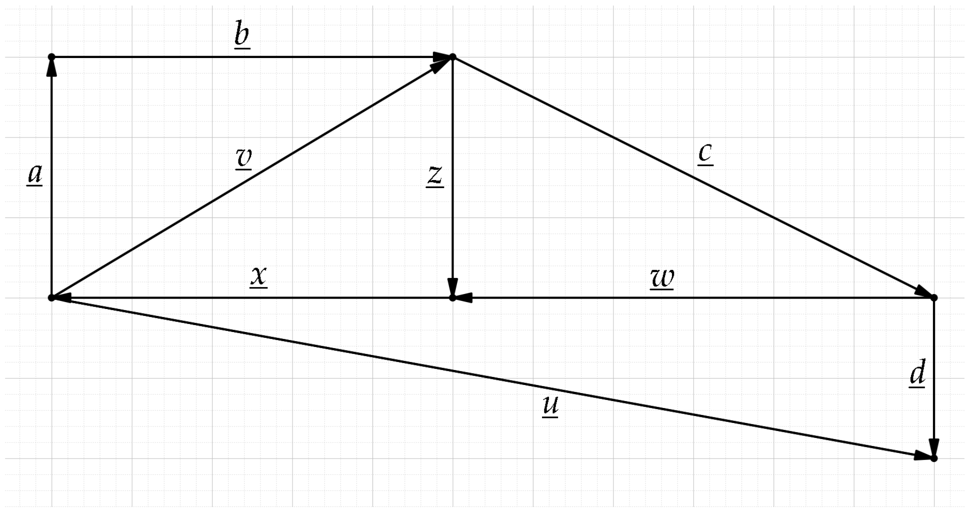
Scalar multiplication has the following properties.

$$0\underline{v} = \underline{0} \qquad -1\underline{v} = -\underline{v} \qquad c(\underline{u} + \underline{v}) = c\underline{u} + c\underline{v}$$

$$1\underline{v} = \underline{v} \qquad (p+q)\,\underline{v} = p\underline{v} + q\underline{v} \qquad p(q\underline{v}) = (pq)\,\underline{v}$$

Practice Problems

1. Based on the picture, determine whether each of the given statements is true or false.



a) $\underline{v} = \underline{a} + \underline{b}$

b) $\underline{v} = \underline{z} + \underline{x}$

c) $\underline{v} = \underline{x} - \underline{z}$

d) $\underline{c} = \underline{d} - \underline{u} + \underline{v}$

e) $\underline{a} = \underline{z}$

f) $\underline{w} + \underline{x} + \underline{u} = \underline{d}$

g) $\underline{d} + \underline{u} = \underline{x} + \underline{w}$

h) $\underline{w} - \underline{c} = \underline{x} - \underline{v}$

i) $\underline{a} + \underline{b} + \underline{z} + \underline{x} = \underline{0}$

j) $\underline{a} + \underline{b} + \underline{c} + \underline{w} + \underline{x} = \underline{0}$

k) $\underline{x} + \underline{w} + \underline{v} = -\underline{c}$

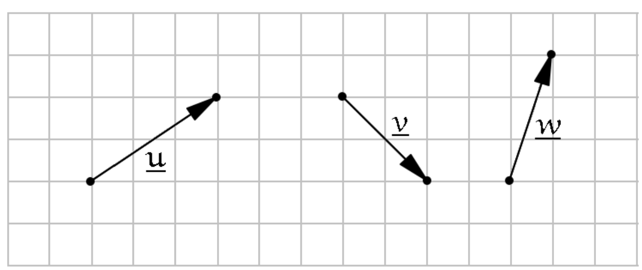
l) $\underline{v} + \underline{c} + \underline{d} = -\underline{u}$

m) $\underline{a} + \underline{z} = \underline{0}$

n) $\underline{a} - \underline{b} = -\underline{v}$

o) $\underline{z} - \underline{w} = \underline{c}$

2. Given the vectors $\underline{u}$, $\underline{v}$ and $\underline{w}$, draw each of the following vectors.

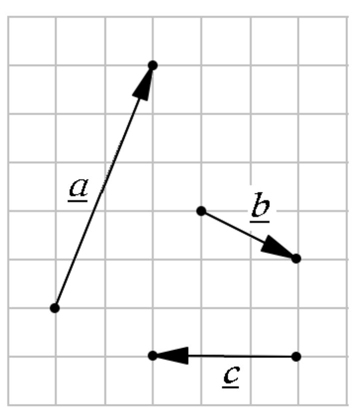


a) $\underline{u} + \underline{v}$

b) $\underline{v} - \underline{w}$

c) $2\underline{u} - \underline{v} + \underline{w}$

3. Given the vectors $\underline{a}$, $\underline{b}$ and $\underline{c}$, draw each of the following vectors.



a) $\underline{a} + \underline{b}$

b) $3\underline{b}$

c) $\underline{b} - \underline{a}$

d) $\underline{a} + \underline{b} + \underline{c}$

e) $\underline{a} - 2\underline{b} + \underline{c}$

f) $-2\underline{c}$

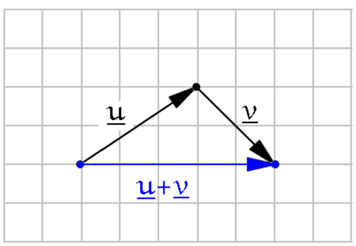
g) $-2\underline{a} + 3\underline{b} - \underline{c}$

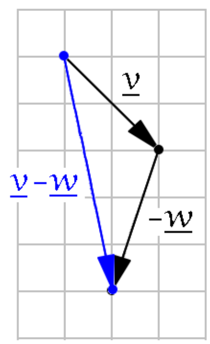
4. Suppose that M is the midpoint of line segment AB. Let O be a point in the plane of AB. Let $\underline{a}$ denote the vector pointing from O to A, let $\underline{b}$ denote the vector pointing from O to B, and $\underline{m}$ denote the vector pointing from O to M. Prove that $\underline{m} = \dfrac{\underline{a}+\underline{b}}{2}$. (Hint: Let $\underline{x}$ be the vector pointing from A to M. Express first $\underline{x}$ in tems of $\underline{a}$ and $\underline{b}$. $\underline{m} = \underline{a} + \underline{x}$).

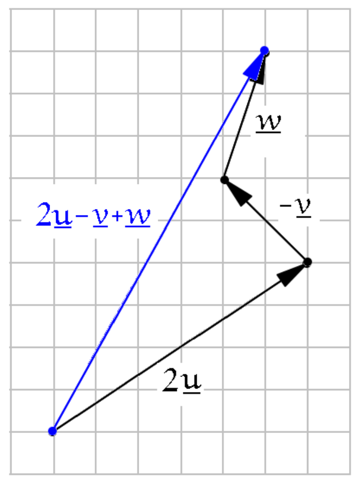
Answers - Practice Problems

1. a) true b) false c) false d) true e) false f) true g) false h) false i) true j) true k) true l) false m) true n) false o) true

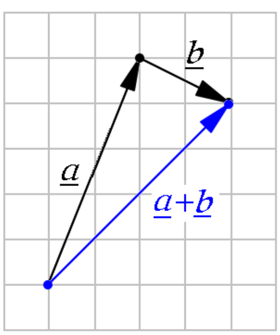
2. a) $\underline{u} + \underline{v}$ b) $\underline{v} - \underline{w}$ c) $2\underline{u} - \underline{v} + \underline{w}$

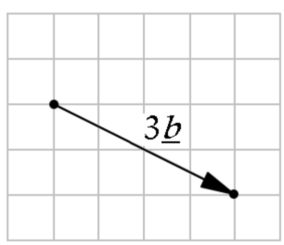


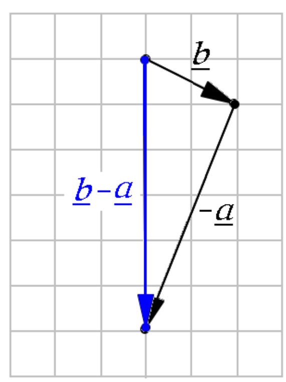




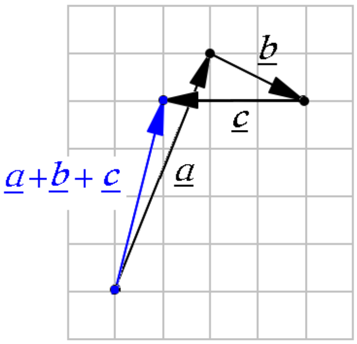
3. a) $\underline{a} + \underline{b}$ b) $3\underline{b}$ c) $\underline{b} - \underline{a}$



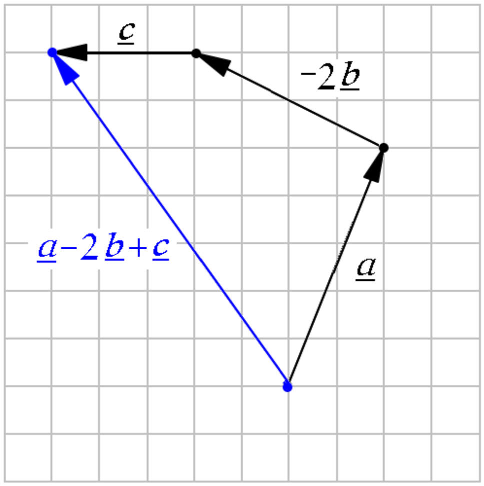




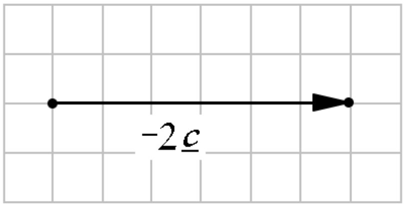
d) $\underline{a}+\underline{b}+\underline{c}$



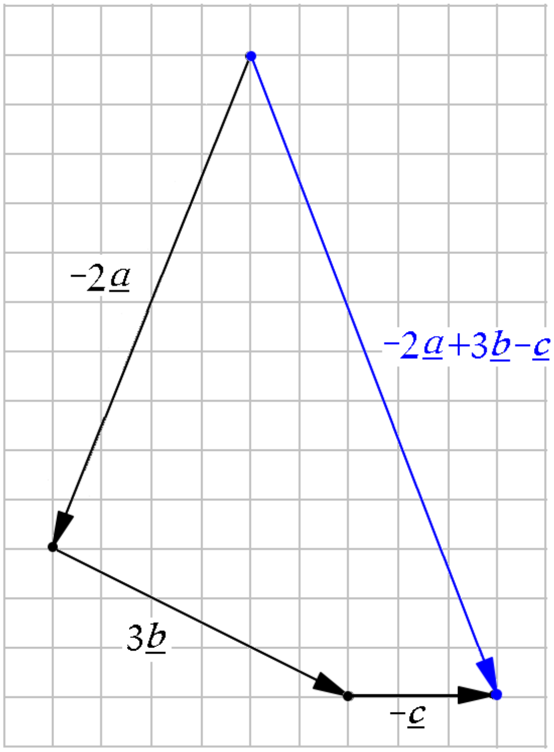
e) $\underline{a}-2\underline{b}+\underline{c}$



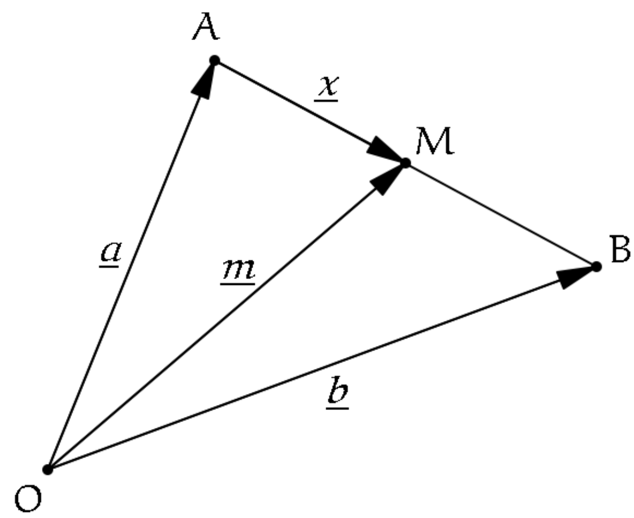
f) $-2\underline{c}$



g) $-2\underline{a}+3\underline{b}-\underline{c}$



4. Suppose that M is the midpoint of line segment AB. Let O be a point in the plane of AB. Let $\underline{a}$ denote the vector pointing from O to A, let $\underline{b}$ denote the vector pointing from O to B, and $\underline{m}$ denote the vector pointing from O to M.

   Claim: $\underline{m} = \dfrac{\underline{a}+\underline{b}}{2}$

   Proof: The vector pointing from A to B is $\underline{b}-\underline{a}$. Then $x = \dfrac{1}{2}(\underline{b}-\underline{a})$.

$$\begin{aligned} \underline{m} &= \underline{a}+\underline{x} = \underline{a}+\frac{1}{2}(\underline{b}-\underline{a}) = \frac{2\underline{a}+\underline{b}-\underline{a}}{2} \\ &= \frac{\underline{a}+\underline{b}}{2} \end{aligned}$$