

We are about to learn about an important integration technique called substitution or u-substitution. We will also see that this technique in practice is a reversal of the chain rule. The following examples can be solved by u-substitution, but they are not complicated enough to really need it. On the other hand, this technique might motivate the idea of substitution. This technique works only for a specific type of a function, namely composed functions in which the inner function is linear.

The technique: Guess the answer. Differentiate your guess. **If we are off only by a constant multiplier**, that can be fixed by a pre-emptive division.

Example 1. Find $\int \cos\left(3x + \frac{\pi}{2}\right) dx$.

Solution: We know that the antiderivative of cosine is sine. So it is a reasonable guess that the answer should be something like $\sin(3x + \pi)$. Let us differentiate our guess.

$$\left(\sin\left(3x + \frac{\pi}{2}\right)\right)' = \cos\left(3x + \frac{\pi}{2}\right) \cdot 3$$

So, the antiderivative is close, but not perfect. Our guess overlooked the multiplier of 3 that pops out of the chain rule when differentiating. We can fix that: our answer should be then $\frac{1}{3} \sin\left(3x + \frac{\pi}{2}\right)$. When we differentiate, we get the derivative $\cos\left(3x + \frac{\pi}{2}\right)$. Therefore, $\int \cos\left(3x + \frac{\pi}{2}\right) dx = \frac{1}{3} \sin\left(3x + \frac{\pi}{2}\right) + C$.

The technique works because derivatives behave nicely with constant multipliers: $(c \cdot f)' = c \cdot f'$ for c real number and f differentiable function. Linear functions are the only functions to have a constant derivative.

Example 2. Find $\int \frac{1}{1-x} dx$.

Solution: The smaller the chain rule, the more dangerous it is. The constant multiplier caused by the chain rule is -1 . It is a common error to overlook it. The antiderivative of $\frac{1}{x}$ is $\ln|x|$. Our first guess will be off by -1 . The correct answer is then $\int \frac{1}{1-x} dx = -\ln|x-1| + C$ or $-\ln|1-x| + C$.

Example 3. Find $\int \frac{6x+1}{2x-3} dx$.

Solution: We can use long division to divide $6x+1$ by $2x-3$. What we will do here, is pretty much the same thing but using a different notation. $\frac{6x}{2x} = 3$, and $3(2x-3) = 6x-9$. But the numerator is $6x+1$. So we will smuggle in what we need for the division.

$$\frac{6x+1}{2x-3} = \frac{6x-9+9+1}{2x-3} = \frac{6x-9}{2x-3} + \frac{10}{2x-3} = 3 + \frac{10}{2x-3}$$

$$\int \frac{6x+1}{2x-3} dx = \int \left(3 + \frac{10}{2x-3}\right) dx = 3x + 10 \cdot \frac{1}{2} \ln|2x-3| + C = 3x + 5 \ln|2x-3| + C$$

Example 4. Find $\int \cos^2 x dx$.

Solution: We will use the double angle formula for cosine. $\cos 2x = 2 \cos^2 x - 1$, then $\cos^2 x = \frac{1 + \cos 2x}{2}$. This will be much easier to integrate.

$$\int \cos^2 x dx = \int \left(\frac{1}{2} + \frac{1}{2} \cos 2x\right) dx = \frac{1}{2}x + \frac{1}{4} \sin 2x + C$$



Practice Problems

Compute each of the following antiderivatives.

1. $\int e^{5x} dx$

6. $\int e^{8t-1} dt$

10. $\int \frac{1}{\left(\frac{1}{3}x - 1\right)^5} dx$

2. $\int \sin 2\theta d\theta$

7. $\int \cos(\pi\theta) d\theta$

11. $\int (-2^{5t} + 1) dt$

3. $\int (2y + 3)^{10} dy$

8. $\int \frac{x+2}{x-1} dx$

12. $\int \sqrt{5m+1} dm$

4. $\int \frac{1}{7x-3} dx$

9. $\int \cos\left(2\alpha - \frac{\pi}{4}\right) d\alpha$

13. $\int \frac{1}{\sqrt{4a+1}} da$



Answers

1. $\frac{1}{5}e^{5x} + C$
2. $-\frac{1}{2}\cos 2\theta + C$
3. $\int \frac{(2y+3)^{11}}{22} + C$
4. $\frac{1}{7}\ln|7x-3| + C$
5. $-\frac{1}{4}(2-x)^4 + C$
6. $\frac{1}{8}e^{8t-1} + C$
7. $\frac{1}{\pi}\sin \pi\theta + C$
8. $x + 3\ln|x-1| + C$
9. $\frac{1}{2}\sin\left(2\alpha - \frac{\pi}{4}\right) + C$
10. $-\frac{3}{4\left(\frac{1}{3}x-1\right)^4} + C$
11. $-\frac{2^{5t}}{5\ln 2} + t + C$
12. $\frac{2}{15}(5m+1)^{3/2} + C$
13. $\frac{1}{2}\sqrt{4a+1} + C$