

Sample Problems

Compute the arc length of the graph of the given function on the interval given.

1. $f(x) = 2(x-1)^{3/2}$ on $[1, 5]$

3. $f(x) = \frac{x^3}{6} + \frac{1}{2x}$ on $[1, 3]$

2. $f(x) = \frac{2}{3}(x^2+1)^{3/2}$ on $[1, 4]$

4. $f(x) = \ln(\cos x)$ on $\left[0, \frac{\pi}{4}\right]$

Practice Problems

Compute the arc length of the graph of the given function on the interval given.

1. $f(x) = \cosh x$ on $[0, 1]$

3. $f(x) = x^2 - \frac{1}{8} \ln x$ on $[1, 2]$

2. $f(x) = \sqrt{4-x^2}$ on $[-2, 2]$

4. $f(x) = x^{3/2}$ on $[0, 1]$

Answers - Sample Problems

1.) $\frac{2}{27}(37\sqrt{37}-1)$ 2.) 45 3.) $\frac{14}{3}$ 4.) $\ln(1+\sqrt{2})$

Answers - Practice Problems

1.) $\frac{e}{2} - \frac{1}{2e}$ 2.) 2π 3.) $3 + \frac{1}{8} \ln 2$ 4.) $\frac{13\sqrt{13}-8}{27}$

Sample Problems - Solutions

Compute the arc length of the graph of the given function on the interval given.

1. $f(x) = 2(x-1)^{3/2}$ on $[1, 5]$

Solution:

$$\begin{aligned} f'(x) &= 2 \left(\frac{3}{2} \right) (x-1)^{1/2} = 3\sqrt{x-1} \\ [f'(x)]^2 &= 9(x-1) = 9x-9 \end{aligned}$$

$$\begin{aligned} L &= \int_1^5 \sqrt{1 + [f'(x)]^2} \, dx = \int_1^5 \sqrt{1 + 9x - 9} \, dx = \int_1^5 (9x - 8)^{1/2} \, dx = \frac{1}{9} \left(\frac{2}{3} \right) (9x - 8)^{3/2} \Big|_1^5 \\ &= \frac{2}{27} \left((9 \cdot 5 - 8)^{3/2} - (9 \cdot 1 - 8)^{3/2} \right) = \frac{2}{27} (37^{3/2} - 1^{3/2}) = \boxed{\frac{2}{27} (37\sqrt{37} - 1)} \end{aligned}$$

2. $f(x) = \frac{2}{3}(x^2+1)^{3/2}$ on $[1, 4]$

Solution:

$$\begin{aligned} f(x) &= \frac{2}{3}(x^2+1)^{3/2} \\ f'(x) &= \frac{2}{3} \left(\frac{3}{2} \right) (x^2+1)^{1/2} (2x) = 2x\sqrt{x^2+1} \\ [f'(x)]^2 + 1 &= 4x^2(x^2+1) + 1 = 4x^4 + 4x^2 + 1 \\ \sqrt{[f'(x)]^2 + 1} &= \sqrt{4x^4 + 4x^2 + 1} = \sqrt{(2x^2+1)^2} = 2x^2 + 1 \end{aligned}$$

$$\begin{aligned} L &= \int_1^4 \sqrt{1 + [f'(x)]^2} \, dx = \int_1^4 2x^2 + 1 \, dx = \frac{2x^3}{3} + x \Big|_1^4 = \left(\frac{2 \cdot 4^3}{3} + 4 \right) - \left(\frac{2 \cdot 1^3}{3} + 1 \right) \\ &= \left(\frac{128}{3} + 4 \right) - \left(\frac{2}{3} + 1 \right) = \frac{126}{3} + 3 = \boxed{45} \end{aligned}$$

3. $f(x) = \frac{x^3}{6} + \frac{1}{2x}$ on $[1, 3]$

Solution:

$$\begin{aligned} f(x) &= \frac{x^3}{6} + \frac{1}{2x} \\ f'(x) &= \frac{3x^2}{6} - \frac{1}{2x^2} = \frac{x^2}{2} - \frac{1}{2x^2} = \frac{1}{2} \left(x^2 - \frac{1}{x^2} \right) \\ [f'(x)]^2 + 1 &= \frac{1}{4} \left(x^4 + \frac{1}{x^4} - 2 \right) + 1 = \frac{1}{4} \left(x^4 + \frac{1}{x^4} - 2 \right) + \frac{4}{4} = \frac{1}{4} \left(x^4 + \frac{1}{x^4} + 2 \right) = \left[\frac{1}{2} \left(x^2 + \frac{1}{x^2} \right) \right]^2 \end{aligned}$$

$$\begin{aligned}
L &= \int_1^3 \sqrt{1 + [f'(x)]^2} \, dx = \int_1^3 \sqrt{\left[\frac{1}{2} \left(x^2 + \frac{1}{x^2}\right)\right]^2} \, dx = \int_1^3 \frac{1}{2} \left(x^2 + \frac{1}{x^2}\right) \, dx = \frac{1}{2} \int_1^3 x^2 + \frac{1}{x^2} \, dx \\
&= \frac{1}{2} \left(\frac{x^3}{3} - \frac{1}{x}\right) \Big|_1^3 = \frac{1}{2} \left[\left(\frac{3^3}{3} - \frac{1}{3}\right) - \left(\frac{1^3}{3} - \frac{1}{1}\right)\right] = \frac{1}{2} \left[\left(9 - \frac{1}{3}\right) - \left(\frac{1}{3} - 1\right)\right] \\
&= \frac{1}{2} \left[\frac{26}{3} - \left(-\frac{2}{3}\right)\right] = \frac{1}{2} \left(\frac{28}{3}\right) = \boxed{\frac{14}{3}}
\end{aligned}$$

4. $f(x) = \ln(\cos x)$ on $\left[0, \frac{\pi}{4}\right]$

Solution:

$$\begin{aligned}
f(x) &= \ln(\cos x) \\
f'(x) &= \frac{1}{\cos x} (-\sin x) = -\tan x \\
[f'(x)]^2 + 1 &= \tan^2 x + 1 = \sec^2 x
\end{aligned}$$

$$\begin{aligned}
L &= \int_0^{\pi/4} \sqrt{1 + [f'(x)]^2} \, dx = \int_0^{\pi/4} \sqrt{\sec^2 x} \, dx = \int_0^{\pi/4} \sec x \, dx = \ln |\sec x + \tan x| \Big|_0^{\pi/4} \\
&= \ln \left| \sec \frac{\pi}{4} + \tan \frac{\pi}{4} \right| - \ln |\sec 0 + \tan 0| = \ln |\sqrt{2} + 1| - \ln |1 + 0| = \ln(\sqrt{2} + 1) - \ln 1 \\
&= \boxed{\ln(\sqrt{2} + 1)}
\end{aligned}$$