

The derivative of a function of a real variable expresses the rate of change of a quantity (a function value or dependent variable) which is determined by another quantity (the independent variable). Derivatives are a fundamental tool of calculus. For example, the derivative of the location function of a moving object with respect to time is the object's velocity: it measures how quickly the position of the object changes when time is advanced.

Suppose that f is a function. Let a be a fixed number in the domain of f .

The expression $\frac{f(a+h) - f(a)}{h}$ is called the **difference quotient**. The difference quotient has a geometric meaning: it is the slope of the secant line connecting two points on the graph of f : $A(a, f(a))$ and $B(a+h, f(a+h))$. The difference quotient also has an interpretation in physics: if the function f is location function, then the difference quotient expresses the average velocity between times $t_1 = a$ and $t_2 = a+h$.

The **derivative** of f , at the number a , denoted by $f'(a)$, is defined as the limit of the difference quotient:

$$f'(a) = \lim_{h \rightarrow 0} \frac{f(a+h) - f(a)}{h}$$

The derivative of f at a is only defined if the limit shown above exists and is finite.

The derivative also has a geometric meaning: $f'(a)$ is the slope of the tangent line drawn to the graph of f at point $(a, f(a))$. The difference quotient also has an interpretation in physics: if the function f is location function, then $f'(a)$ expresses the instantaneous velocity at $t = a$.

Given a function f , if we evaluate $f'(x)$ for all x , we obtain a new function, called the derivative (or first derivative) of f .

Differentiate each of the following by evaluating the limit of the difference quotient.

1. $f(x) = x^2 - 3x$

3. $f(x) = \sqrt{2x-1}$

5. $f(x) = \sqrt{1-x^2}$

2. $f(x) = x^3$

4. $f(x) = \frac{1}{x^2-1}$

Solutions

1. $f(x) = x^2 - 3x$

$$\begin{aligned} f'(x) &= \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \rightarrow 0} \frac{((x+h)^2 - 3(x+h)) - (x^2 - 3x)}{h} \\ &= \lim_{h \rightarrow 0} \frac{(x^2 + 2xh + h^2 - 3x - 3h) - (x^2 - 3x)}{h} = \lim_{h \rightarrow 0} \frac{x^2 + 2xh + h^2 - 3x - 3h - x^2 + 3x}{h} \\ &= \lim_{h \rightarrow 0} \frac{2xh + h^2 - 3h}{h} = \lim_{h \rightarrow 0} \frac{h(2x + h - 3)}{h} = \lim_{h \rightarrow 0} (2x + h - 3) = 2x - 3 \end{aligned}$$

2. $f(x) = x^3$

$$\begin{aligned} f'(x) &= \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \rightarrow 0} \frac{(x+h)^3 - x^3}{h} = \lim_{h \rightarrow 0} \frac{x^3 + 3x^2h + 3xh^2 + h^3 - x^3}{h} \\ &= \lim_{h \rightarrow 0} \frac{3x^2h + 3xh^2 + h^3}{h} = \lim_{h \rightarrow 0} \frac{h(3x^2 + 3xh + h^2)}{h} = \lim_{h \rightarrow 0} 3x^2 + 3xh + h^2 = 3x^2 \end{aligned}$$

3. $f(x) = \sqrt{2x-1}$

$$\begin{aligned} f'(x) &= \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \rightarrow 0} \frac{\sqrt{2(x+h)-1} - \sqrt{2x-1}}{h} \\ &= \lim_{h \rightarrow 0} \left(\frac{\sqrt{2x+2h-1} - \sqrt{2x-1}}{h} \cdot \frac{\sqrt{2x+2h-1} + \sqrt{2x-1}}{\sqrt{2x+2h-1} + \sqrt{2x-1}} \right) \\ &= \lim_{h \rightarrow 0} \frac{(2x+2h-1) - (2x-1)}{h(\sqrt{2x+2h-1} + \sqrt{2x-1})} = \lim_{h \rightarrow 0} \frac{2x+2h-1-2x+1}{h(\sqrt{2x+2h-1} + \sqrt{2x-1})} \\ &= \lim_{h \rightarrow 0} \frac{2h}{h(\sqrt{2x+2h-1} + \sqrt{2x-1})} = \lim_{h \rightarrow 0} \frac{2}{\sqrt{2x+2h-1} + \sqrt{2x-1}} = \frac{2}{2\sqrt{2x-1}} = \frac{1}{\sqrt{2x-1}} \end{aligned}$$

4. $f(x) = \frac{1}{x^2-1}$

$$\begin{aligned} f'(x) &= \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \rightarrow 0} \frac{\frac{1}{(x+h)^2-1} - \frac{1}{x^2-1}}{h} \\ &= \lim_{h \rightarrow 0} \frac{\frac{x^2-1}{((x+h)^2-1)(x^2-1)} - \frac{(x+h)^2-1}{(x^2-1)((x+h)^2-1)}}{h} = \lim_{h \rightarrow 0} \left(\frac{x^2-1 - ((x+h)^2-1)}{((x+h)^2-1)(x^2-1)} \cdot \frac{1}{h} \right) \\ &= \lim_{h \rightarrow 0} \frac{x^2-1 - (x^2+2xh+h^2-1)}{h((x+h)^2-1)(x^2-1)} = \lim_{h \rightarrow 0} \left(\frac{x^2-1-x^2-2xh-h^2+1}{h((x+h)^2-1)(x^2-1)} \right) \\ &= \lim_{h \rightarrow 0} \frac{-2xh-h^2}{h((x+h)^2-1)(x^2-1)} = \lim_{h \rightarrow 0} \frac{-h(2x+h)}{h((x+h)^2-1)(x^2-1)} = \lim_{h \rightarrow 0} \frac{-(2x+h)}{((x+h)^2-1)(x^2-1)} \\ &= \frac{-2x}{(x^2-1)(x^2-1)} = \frac{-2x}{(x^2-1)^2} \end{aligned}$$

5. $f(x) = \sqrt{1-x^2}$

$$\begin{aligned}
 f'(x) &= \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \rightarrow 0} \frac{\sqrt{1-(x+h)^2} - \sqrt{1-x^2}}{h} \\
 &= \lim_{h \rightarrow 0} \frac{\sqrt{1-(x+h)^2} - \sqrt{1-x^2}}{h} \cdot \frac{\sqrt{1-(x+h)^2} + \sqrt{1-x^2}}{\sqrt{1-(x+h)^2} + \sqrt{1-x^2}} \\
 &= \lim_{h \rightarrow 0} \frac{1-(x+h)^2 - (1-x^2)}{h \left(\sqrt{1-(x+h)^2} + \sqrt{1-x^2} \right)} = \lim_{h \rightarrow 0} \frac{1-x^2-h^2-2xh-1+x^2}{h \left(\sqrt{1-(x+h)^2} + \sqrt{1-x^2} \right)} \\
 &= \lim_{h \rightarrow 0} \frac{-h^2-2xh}{h \left(\sqrt{1-(x+h)^2} + \sqrt{1-x^2} \right)} = \lim_{h \rightarrow 0} \frac{-h(h+2x)}{h \left(\sqrt{1-(x+h)^2} + \sqrt{1-x^2} \right)} \\
 &= \lim_{h \rightarrow 0} \frac{-(h+2x)}{\sqrt{1-(x+h)^2} + \sqrt{1-x^2}} = \frac{-2x}{\sqrt{1-x^2} + \sqrt{1-x^2}} = \frac{-2x}{2\sqrt{1-x^2}} = \frac{-x}{\sqrt{1-x^2}}
 \end{aligned}$$

6. $f(x) = \frac{1}{x+2}$

$$\begin{aligned}
 f'(x) &= \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \rightarrow 0} \frac{\frac{1}{x+h+2} - \frac{1}{x+2}}{h} = \lim_{h \rightarrow 0} \frac{\frac{x+2}{(x+h+2)(x+2)} - \frac{x+h+2}{(x+h+2)(x+2)}}{h} \\
 &= \lim_{h \rightarrow 0} \frac{1}{h} \cdot \frac{x+2-x-h-2}{(x+h+2)(x+2)} = \lim_{h \rightarrow 0} \frac{1}{h} \cdot \frac{-h}{(x+h+2)(x+2)} = \lim_{h \rightarrow 0} \frac{-1}{(x+h+2)(x+2)} = -\frac{1}{(x+2)^2}
 \end{aligned}$$

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