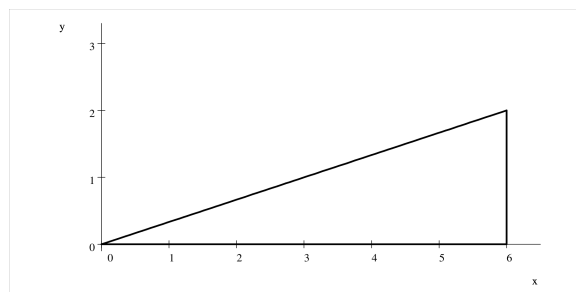


One dimension

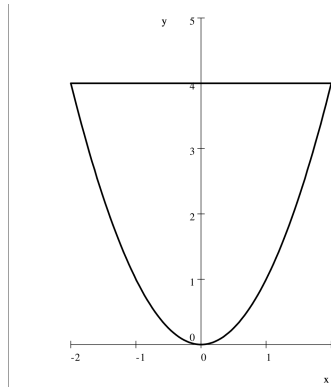
1. Compute the center of mass of the system consisting of m_1 and m_2 located on the number line if given that $m_1 = 3$ g at $x_1 = 2$ and $m_2 = 7$ g at $x_2 = 6$.
2. Compute the center of mass of the system consisting of m_1 , m_2 , and m_3 located on the number line if given that $m_1 = 5$ g at $x_1 = -2$, $m_2 = 8$ g at $x_2 = 1$, and $m_3 = 3$ g at $x_3 = 5$.
3. Compute the center of mass of the system consisting of m_1 , m_2 , and m_3 located on the number line if given that $m_1 = 1$ g at $x_1 = 1$, $m_2 = 2$ g at $x_2 = 2$, and $m_3 = 3$ g at $x_3 = 3$.
4. Compute the center of mass of the system consisting of $m_1, m_2, \dots, m_n$ located on the number line at $x_1, x_2, \dots, x_n$.
5. A rod of length 3 units is made of material that has different density at different locations. If the rod is located at such that its ends are at $(0, 0)$ and $(3, 0)$, then the density of the material at x is given by $\delta(x) = x^2 \frac{\text{g}}{\text{cm}}$.
 - a) Compute the mass of the rod.
 - b) Compute the center of mass of the rod.
6. A rod of length 5 units is made of material that has different density at different locations. If the rod is located at such that its ends are at $(1, 0)$ and $(6, 0)$, then the density of the material at x is given by $\delta(x) = \frac{1}{x^2} \frac{\text{g}}{\text{cm}}$.
 - a) Compute the mass of the rod.
 - b) Compute the center of mass of the rod.

Two dimensions

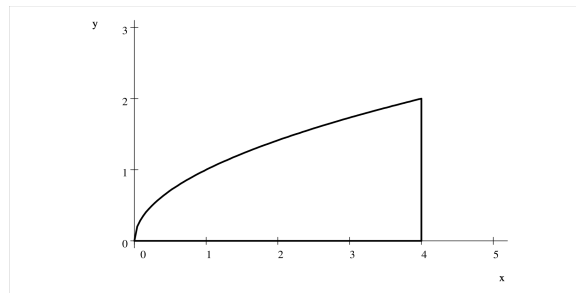
7. Compute the center of mass of the system consisting of m_1 and m_2 located on the number line if given that $m_1 = 13$ g at $P_1(2, -5)$ and $m_2 = 7$ g at $P_2(-3, 4)$.
8. Compute the center of mass of the system consisting of m_1 , m_2 , and m_3 located on the number line if given that $m_1 = 3$ g at $P_1(-1, 1)$, $m_2 = 5$ g at $P_2(4, -3)$, and $m_3 = 7$ g at $P_3(-2, 0)$.
9. The triangular plate shown on the picture below is bounded by the graphs of $y = \frac{x}{3}$, $y = 0$, and $x = 6$ has a constant density of $5 \frac{\text{g}}{\text{cm}^2}$.



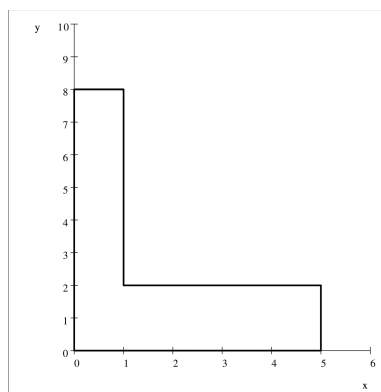
- a) Compute the plate's moment M_x about the y -axis.
 - b) Compute the mass of the plate.
 - c) Compute the x -coordinate of the center of mass of the plate.
 - d) Compute the plate's moment M_y about the x -axis.
 - e) Compute the y -coordinate of the center of mass of the plate.
10. Compute the center of mass of a thin plate bounded by the graphs of $y = x^2$ and $y = 4$ between $x = -2$ and $x = 2$.



11. Compute the center of mass of the region bounded by the graphs of $y = \sqrt{x}$, $y = 0$, and $x = 4$.



12. Compute the center of mass of the region bounded by the graphs of $y = \sqrt{x}$, $y = 0$, and $x = 9$.
13. Compute the center of mass of the object shown on the picture below.



14. Compute the center of mass of the region bounded by the graphs of $y = \frac{1}{x^3}$, $x = 1$, and $y = 0$.

Answers

1. 4.8

2. $\frac{13}{16} = 0.8125$

3. $\frac{7}{3}$

4.
$$\frac{\sum_{k=1}^n m_k x_k}{\sum_{k=1}^n m_k}$$

5. a) 9 b) $\frac{9}{4} = 2.25$

6. a) $\frac{5}{6}$ b) $\frac{6}{5} \ln 6$

7. $C\left(\frac{1}{4}, -\frac{37}{20}\right)$

8. $C\left(\frac{1}{5}, -\frac{4}{5}\right)$

9. a) 120 b) 30 c) 4 d) 20 e) $\frac{2}{3}$

10. $\left(0, \frac{12}{5}\right) = (0, 2.4)$

11. $\left(\frac{12}{5}, \frac{3}{4}\right) = (2.4, 0.75)$

12. $\left(\frac{27}{5}, \frac{9}{8}\right) = (5.4, 1.125)$

13. $\left(\frac{7}{4}, \frac{5}{2}\right) = (1.75, 2.5)$

14. $\left(2, \frac{1}{5}\right) = (2, 0.2)$

Solutions - One dimension

1. Compute the center of mass of the system consisting of m_1 and m_2 located on the number line if given that $m_1 = 3$ g at $x_1 = 2$ and $m_2 = 7$ g at $x_2 = 6$.

$$\begin{aligned}
 m_1(x - x_1) + m_2(x - x_2) &= 0 \\
 m_1x - m_1x_1 + m_2x - m_2x_2 &= 0 \\
 m_1x + m_2x &= m_1x_1 + m_2x_2 \\
 x(m_1 + m_2) &= m_1x_1 + m_2x_2 \\
 x &= \frac{m_1x_1 + m_2x_2}{m_1 + m_2} = \frac{3 \cdot 2 + 7 \cdot 6}{3 + 7} = \frac{48}{10} = 4.8
 \end{aligned}$$

2. Compute the center of mass of the system consisting of m_1 , m_2 , and m_3 located on the number line if given that $m_1 = 5$ g at $x_1 = -2$, $m_2 = 8$ g at $x_2 = 1$, and $m_3 = 3$ g at $x_3 = 5$.

Let x be the center of mass.

$$\begin{aligned}
 m_1(x - x_1) + m_2(x - x_2) + m_3(x - x_3) &= 0 \\
 m_1x - m_1x_1 + m_2x - m_2x_2 + m_3x - m_3x_3 &= 0 \\
 m_1x + m_2x + m_3x &= m_1x_1 + m_2x_2 + m_3x_3 \\
 x(m_1 + m_2 + m_3) &= m_1x_1 + m_2x_2 + m_3x_3 \\
 x &= \frac{m_1x_1 + m_2x_2 + m_3x_3}{m_1 + m_2 + m_3}
 \end{aligned}$$

$$x = \frac{m_1x_1 + m_2x_2 + m_3x_3}{m_1 + m_2 + m_3} = \frac{5(-2) + 8 \cdot 1 + 3 \cdot 5}{5 + 8 + 3} = \frac{13}{16}$$

3. Compute the center of mass of the system consisting of m_1 , m_2 , and m_3 located on the number line if given that $m_1 = 1$ g at $x_1 = 1$, $m_2 = 2$ g at $x_2 = 2$, and $m_3 = 3$ g at $x_3 = 3$. $\frac{1^2 + 2^2 + 3^2}{1 + 2 + 3} = \frac{7}{3}$

4. Compute the center of mass of the system consisting of $m_1, m_2, \dots, m_n$ located on the number line at $x_1,$

$$x_2, \dots, x_n. \quad \frac{\sum_{k=1}^n m_k x_k}{\sum_{k=1}^n m_k}$$

5. A rod of length 3 units is made of material that has different density at different locations. If the rod is located at such that its ends are at $(0,0)$ and $(3,0)$, then the density of the material at x is given by $\delta(x) = x^2 \frac{\text{g}}{\text{cm}}$.

a) Compute the mass of the rod. $m = \int_0^3 dm = \int_0^3 \delta(x) dx = \int_0^3 x^2 dx = 9$

b) Compute the center of mass of the rod. $\frac{\int_0^3 x dm}{\int_0^3 dm} = \frac{\int_0^3 x \delta(x) dx}{\int_0^3 \delta(x) dx} = \frac{\int_0^3 x \cdot x^2 dx}{9} = \frac{\frac{81}{4}}{9} = \frac{9}{4}$

6. A rod of length 5 units is made of material that has different density at different locations. If the rod is located at such that its ends are at $(1, 0)$ and $(6, 0)$, then the density of the material at x is given by $\delta(x) = \frac{1}{x^2} \frac{\text{g}}{\text{cm}}$.

a) Compute the mass of the rod.
$$m = \int_1^6 dm = \int_1^6 \delta(x) dx = \int_1^6 \frac{1}{x^2} dx = \frac{5}{6}$$

b) Compute the center of mass of the rod.
$$\frac{\int_1^6 x dm}{\int_1^6 dm} = \frac{\int_1^6 x \delta(x) dx}{\int_1^6 \delta(x) dx} = \frac{\int_1^6 x \frac{1}{x^2} dx}{\frac{5}{6}} = \frac{6}{5} \ln 6$$

Two dimensions

7. Compute the center of mass of the system consisting of m_1 and m_2 located on the number line if given that $m_1 = 13 \text{ g}$ at $P_1(2, -5)$ and $m_2 = 7 \text{ g}$ at $P_2(-3, 4)$.

$$\bar{x} = \frac{m_1 x_1 + m_2 x_2}{m_1 + m_2} = \frac{13 \cdot 2 + 7 \cdot (-3)}{13 + 7} = \frac{1}{4}$$

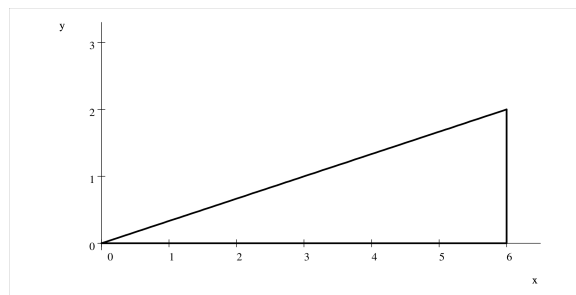
$$\bar{y} = \frac{m_1 y_1 + m_2 y_2}{m_1 + m_2} = \frac{13(-5) + 7 \cdot (4)}{13 + 7} = -\frac{37}{20} \quad C\left(\frac{1}{4}, -\frac{37}{20}\right)$$

8. Compute the center of mass of the system consisting of m_1 , m_2 , and m_3 located on the number line if given that $m_1 = 3 \text{ g}$ at $P_1(-1, 1)$, $m_2 = 5 \text{ g}$ at $P_2(4, -3)$, and $m_3 = 7 \text{ g}$ at $P_3(-2, 0)$.

$$\bar{x} = \frac{m_1 x_1 + m_2 x_2 + m_3 x_3}{m_1 + m_2 + m_3} = \frac{3(-1) + 5 \cdot 4 + 7(-2)}{3 + 5 + 7} = \frac{1}{5}$$

$$\bar{y} = \frac{m_1 y_1 + m_2 y_2 + m_3 y_3}{m_1 + m_2 + m_3} = \frac{3 \cdot 1 + 5(-3) + 7 \cdot 0}{3 + 5 + 7} = -\frac{4}{5} \quad C\left(\frac{1}{5}, -\frac{4}{5}\right)$$

9. The triangular plate shown on the picture below is bounded by the graphs of $y = \frac{x}{3}$, $y = 0$, and $x = 6$ has a constant density of $5 \frac{\text{g}}{\text{cm}^2}$.



a) Compute the plate's moment M_x about the y -axis.

Let us first slice the object into very thin vertical slices. In a general slice,

center of mass: $(\tilde{x}, \tilde{y}) = \left(x, \frac{x}{6}\right)$

vertical side: $\frac{x}{3}$

horizontal side: dx area: $dA = \frac{x}{3}dx$

mass: $dm = \delta dA = 5 \cdot \frac{x}{3}dx = \frac{5x}{3}dx$

The moment of the strip about the y -axis: $\tilde{x}dm = x \left(\frac{5x}{3}dx\right) = \frac{5}{3}x^2dx$

The moment of the plate about the y -axis: $\int \tilde{x}dm = \int_0^6 \frac{5}{3}x^2dx = 120$

b) Compute the mass of the plate. $M = \int dm = \int_0^6 \frac{5}{3}x dx = 30$

c) Compute the x -coordinate of the center of mass of the plate. $\bar{x} = \frac{\int \tilde{x}dm}{\int dm} = \frac{120}{30} = 4$

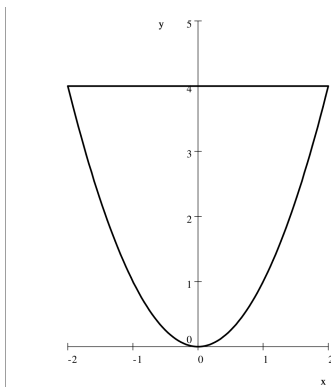
d) Compute the plate's moment M_y about the x -axis.

The moment of the strip about the x -axis: $\tilde{y}dm = \frac{x}{6} \left(\frac{5x}{3}dx\right) = \frac{5}{18}x^2dx$

The moment of the plate about the x -axis: $\int \tilde{y}dm = \int_0^6 \frac{5}{18}x^2dx = 20$

e) Compute the y -coordinate of the center of mass of the plate. $\bar{y} = \frac{\int \tilde{y}dm}{\int dm} = \frac{20}{30} = \frac{2}{3}$

10. Compute the center of mass of a thin plate bounded by the graphs of $y = x^2$ and $y = 4$ between $x = -2$ and $x = 2$.



Solution: a general stripe located at x

center of mass: $(\tilde{x}, \tilde{y}) = \left(x, \frac{1}{2}(x^2 + 4)\right)$

horizontal side: dx

vertical side: $4 - x^2$

area: $dA = (4 - x^2) dx$

mass: $dm = \delta (4 - x^2) dx$

total mass of the plate: $\int dm = \int_{-2}^2 \delta (4 - x^2) dx = \frac{32}{3} \delta$

moment of stripe about the y -axis: $\tilde{y} dm = x \delta (4 - x^2) dx$

moment of plate about the y -axis: $\int \tilde{x} dm = \int_{-2}^2 \delta x (4 - x^2) dx = \delta \int_{-2}^2 x (4 - x^2) dx = 0$

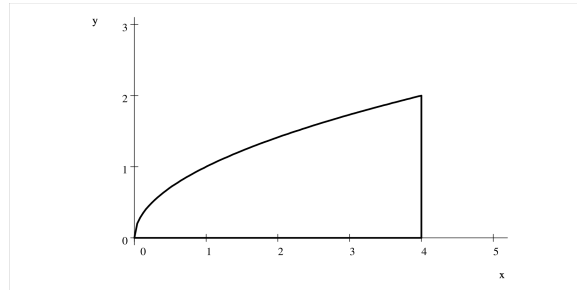
the x -coordinate of center of mass: $\bar{x} = \frac{\int \tilde{x} dm}{\int dm} = \frac{0}{\frac{32}{3} \delta} = 0$

moment of stripe about the x -axis: $\tilde{y} dm = \frac{1}{2} (x^2 + 4) \delta (4 - x^2) dx$

moment of plate about the x -axis: $\int \tilde{y} dm = \int_{-2}^2 \frac{\delta}{2} (x^2 + 4) (4 - x^2) dx = \frac{\delta}{2} \int_{-2}^2 (x^2 + 4) (4 - x^2) dx = \frac{128}{5} \delta$

the y -coordinate of center of mass: $\bar{y} = \frac{\int \tilde{y} dm}{\int dm} = \frac{\frac{128}{5} \delta}{\frac{32}{3} \delta} = \frac{12}{5}$

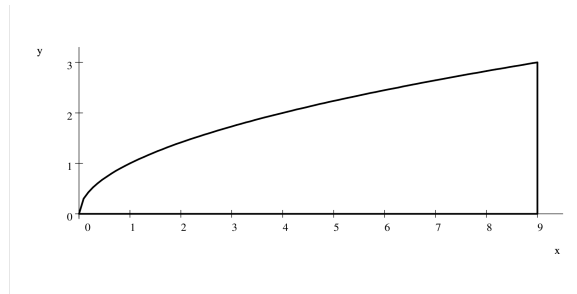
11. Compute the center of mass of the region bounded by the graphs of $y = \sqrt{x}$, $y = 0$, and $x = 4$.



x -coordinate: $\bar{x} = \frac{\int_0^4 x \delta \sqrt{x} dx}{\int_0^4 \delta \sqrt{x} dx} = \frac{\frac{64}{5} \delta}{\frac{16}{3} \delta} = \frac{12}{5} = 2.4$

y -coordinate: $\bar{y} = \frac{\int_0^4 \left(\frac{\sqrt{x}}{2} \right) \delta \sqrt{x} dx}{\int_0^4 \delta \sqrt{x} dx} = \frac{\frac{4\delta}{3}}{\frac{16}{3} \delta} = \frac{3}{4} = 0.75$ $\left(\frac{12}{5}, \frac{3}{4} \right) = (2.4, 0.75)$

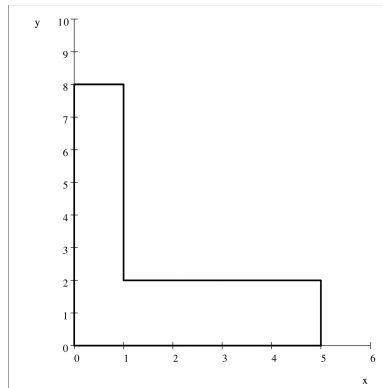
12. Compute the center of mass of the region bounded by the graphs of $y = \sqrt{x}$, $y = 0$, and $x = 9$.



$$x\text{-coordinate: } \bar{x} = \frac{\int_0^9 x\delta\sqrt{x}dx}{\int_0^9 \delta\sqrt{x}dx} = \frac{\frac{486}{5}\delta}{18\delta} = \frac{27}{5} = 5.4$$

$$y\text{-coordinate: } \bar{y} = \frac{\int_0^9 \left(\frac{\sqrt{x}}{2}\right)\delta\sqrt{x}dx}{\int_0^9 \delta\sqrt{x}dx} = \frac{\frac{81}{4}\delta}{18\delta} = \frac{9}{8} = 1.125 \quad \left(\frac{27}{5}, \frac{9}{8}\right) = (5.4, 1.125)$$

13. Compute the center of mass of the object shown on the picture below.



Part 1.

Vertical stripe: dx by 8

mass: $8\delta dx$

$$\text{mass: } \int_0^1 8\delta dx = 8\delta$$

$$\text{moment about the } y\text{-axis: } \int_0^1 8x\delta dx = 4\delta$$

$$\text{moment about the } x\text{-axis: } \int_0^1 8(4)\delta dx = 32\delta$$

Part 2:

Vertical stripe: dx by 2

mass: $2\delta dx$

$$\text{mass: } \int_1^5 2\delta dx = 8\delta$$

$$\text{moment about the } y\text{-axis: } \int_1^5 2x\delta dx = 24\delta$$

$$\text{moment about the } x\text{-axis: } \int_1^5 1(2)\delta dx = 8\delta$$

$$x\text{-coordinate of center of mass: } \frac{\int_0^1 8x\delta dx + \int_1^5 2x\delta dx}{\int_0^1 8\delta dx + \int_1^5 2\delta dx} = \frac{4\delta + 24\delta}{8\delta + 8\delta} = \frac{7}{4}$$

$$y\text{-coordinate of center of mass: } \frac{\int_0^1 32\delta dx + \int_1^5 2\delta dx}{\int_0^1 8\delta dx + \int_1^5 2\delta dx} = \frac{32\delta + 8\delta}{8\delta + 8\delta} = \frac{5}{2} \quad C\left(\frac{7}{4}, \frac{5}{2}\right)$$

14. Compute the center of mass of the region bounded by the graphs of $y = \frac{1}{x^3}$, $x = 1$, and $y = 0$.

$$\bar{x} = \frac{\int_1^\infty x \frac{1}{x^3} dx}{\int_1^\infty \frac{1}{x^3} dx} = 2 \quad \bar{y} = \frac{\int_1^\infty \left(\frac{1}{2x^3}\right) \frac{1}{x^3} dx}{\int_1^\infty \frac{1}{x^3} dx} = \frac{1}{5} \quad \left(2, \frac{1}{5}\right)$$

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