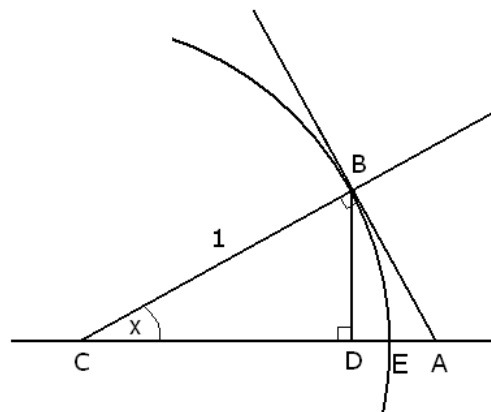


Theorem 1: $\lim_{x \rightarrow 0} \frac{\sin x}{x} = 1$

Proof: This theorem and the next one are necessary for differentiating $\sin x$ and $\cos x$. Recall a theorem: Let r be the radius of a circle. If α is measured in radians, then the area of a sector with a central angle of α is $A_{\text{sector}} = \frac{\alpha r^2}{2}$. (Notation: $\overline{AB}$ will denote the length of line segment AB .)

Let x be a very small positive angle, measured in radians, drawn into a unit circle as shown on the picture below. Let B be the point where the unit circle intersects the ray determined by x . We then draw a tangent line to the circle at point B . Let A be the point where the tangent line intersects the x -axis. We also draw a vertical line through B . Let D be the point where this vertical line intersects the x -axis. Finally, let us denote by E the point with coordinates $(0, 1)$



The proof will be based on the following fact: because they include each other, the following three areas can be easily compared:

$$\text{Area of triangle } CDB \leq \text{Area of sector } CEB \leq \text{Area of triangle } ABC$$

Area of triangle CDB : the horizontal side, $\overline{CD} = \cos x$ and the vertical side, $\overline{DB} = \sin x$. Since this is a right triangle, the area is: $A_{CDB} = \frac{1}{2} \sin x \cos x$

Area of sector CEB : $A_{\text{sector}} = \frac{1^2 x}{2} = \frac{x}{2}$

Area of triangle ABC : there is a right angle at point B because the tangent line drawn to a circle is perpendicular to the radius drawn to the point of tangency. So the area is $A_{ABC} = \frac{1}{2} \overline{AB} \cdot \overline{BC}$. Clearly $\overline{BC} = 1$. To compute $\overline{AB}$, in triangle ABC , $\tan x = \frac{\overline{AB}}{1}$ and so $\overline{AB} = \tan x$.

Area of triangle ABC : $\frac{1}{2} (1) (\tan x) = \frac{\tan x}{2}$ or $\frac{\sin x}{2 \cos x}$. So now

$$\text{Area of triangle } CDB \leq \text{Area of sector } CEB \leq \text{Area of triangle } ABC$$

translates to

$$\frac{1}{2} \sin x \cos x \leq \frac{x}{2} \leq \frac{\sin x}{2 \cos x}$$

Let us divide all three sides by $\frac{\sin x}{2}$. Because x is small and positive, $\frac{\sin x}{2}$ is positive and so we do not need to reverse the inequality signs.

$$\cos x \leq \frac{x}{\sin x} \leq \frac{1}{\cos x}$$

Suppose now that x approaches zero. Then both $\cos x$ and $\frac{1}{\cos x}$ approach 1. By the sandwich principle, $\frac{x}{\sin x}$, the quantity locked in between those two must also approach 1.

$$\begin{array}{ccccc} \cos x & \leq & \frac{x}{\sin x} & \leq & \frac{1}{\cos x} \\ \downarrow & & & & \downarrow \\ 1 & & & & 1 \end{array}$$

If $\frac{x}{\sin x}$ approaches 1, so does its reciprocal, $\frac{\sin x}{x}$.

So far, we have proven the statement for positive values of x , that is, $\lim_{x \rightarrow 0^+} \frac{\sin x}{x} = 1$. A similar argument works for negative values of x .

Theorem 2: $\lim_{x \rightarrow 0} \frac{\cos x - 1}{x} = 0$
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Proof:

$$\begin{aligned} \lim_{x \rightarrow 0} \frac{\cos x - 1}{x} &= \lim_{x \rightarrow 0} \frac{\cos x - 1}{x} \cdot 1 = \lim_{x \rightarrow 0} \left(\frac{\cos x - 1}{x} \cdot \frac{\cos x + 1}{\cos x + 1} \right) = \lim_{x \rightarrow 0} \frac{\cos^2 x - 1}{x(\cos x + 1)} = \lim_{x \rightarrow 0} \frac{-(1 - \cos^2 x)}{x(\cos x + 1)} \\ &= \lim_{x \rightarrow 0} \frac{-\sin^2 x}{x(\cos x + 1)} = \lim_{x \rightarrow 0} \frac{\sin x}{x} \cdot \frac{-\sin x}{\cos x + 1} = \lim_{x \rightarrow 0} \frac{\sin x}{x} \cdot \lim_{x \rightarrow 0} \frac{-\sin x}{\cos x + 1} = 1 \cdot 0 = 0 \end{aligned}$$

We are now ready to prove that $\frac{d}{dx} \sin x = \cos x$ and $\frac{d}{dx} \cos x = -\sin x$

Theorems 3 and 4: $\frac{d}{dx} \sin x = \cos x$ and $\frac{d}{dx} \cos x = -\sin x$
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Proof:

$$\begin{aligned} \frac{d}{dx} \sin x &= \lim_{h \rightarrow 0} \frac{\sin(x+h) - \sin x}{h} = \lim_{h \rightarrow 0} \frac{\sin x \cos h + \cos x \sin h - \sin x}{h} \\ &= \lim_{h \rightarrow 0} \left(\frac{\sin x \cos h - \sin x}{h} + \frac{\cos x \sin h}{h} \right) = \lim_{h \rightarrow 0} \frac{\sin x (\cos h - 1)}{h} + \lim_{h \rightarrow 0} \cos x \frac{\sin h}{h} \\ &= \sin x \lim_{h \rightarrow 0} \frac{\cos h - 1}{h} + \cos x \lim_{h \rightarrow 0} \frac{\sin h}{h} = \sin x \cdot 0 + \cos x \cdot 1 = \cos x \end{aligned}$$

$$\begin{aligned} \frac{d}{dx} \cos x &= \lim_{h \rightarrow 0} \frac{\cos(x+h) - \cos x}{h} = \lim_{h \rightarrow 0} \frac{\cos x \cos h - \sin x \sin h - \cos x}{h} \\ &= \lim_{h \rightarrow 0} \left(\frac{\cos x \cos h - \cos x}{h} - \frac{\sin x \sin h}{h} \right) = \lim_{h \rightarrow 0} \frac{\cos x (\cos h - 1)}{h} - \lim_{h \rightarrow 0} \frac{\sin x \sin h}{h} \\ &= \cos x \lim_{h \rightarrow 0} \frac{\cos h - 1}{h} - \sin x \lim_{h \rightarrow 0} \frac{\sin h}{h} = \cos x \cdot 0 - \sin x \cdot 1 = -\sin x \end{aligned}$$

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