

$f : (a, b) \rightarrow \mathbb{R}$ is convex if for all $x, y \in (a, b)$ and $0 < \lambda < 1$ we have

$$f(\lambda x + (1 - \lambda)y) \leq \lambda f(x) + (1 - \lambda)f(y)$$

1. Prove that if f is convex on (a, b) , then it is continuous at every point of (a, b) .
2. Prove that if $w < x < y < z$ for $w, x, y, z \in (a, b)$, then

$$\frac{f(x) - f(w)}{x - w} \leq \frac{f(z) - f(y)}{z - y}$$

3. Suppose that $g : (a, b) \rightarrow \mathbb{R}$ is differentiable. Prove that g is convex on (a, b) if and only if $g' : (a, b) \rightarrow \mathbb{R}$ is a non-decreasing function.
4. Suppose that $h : (a, b) \rightarrow \mathbb{R}$ is twice differentiable (so h'' exists for all $x \in (a, b)$). Prove that h is convex on (a, b) if and only if $h''(x) \geq 0$ for all $x \in (a, b)$.

Let $w < x < y < z$ for $w, x, y, z \in (a, b)$. Define $\lambda = \frac{z - x}{z - w}$. Then $1 - \lambda = \frac{x - w}{z - w}$, since

$$\frac{z - x}{z - w} + \frac{x - w}{z - w} = 1$$

Also, both λ and $1 - \lambda$ are between 0 and 1. The expression $f(\lambda w + (1 - \lambda)z)$ becomes

$$f(\lambda w + (1 - \lambda)z) = f\left(\frac{z - x}{z - w}w + \frac{x - w}{z - w}z\right) = f\left(\frac{zw - xw + xz - wz}{z - w}\right) = f\left(\frac{xz - xw}{z - w}\right) = f(x)$$

And so the statement

$$f(\lambda w + (1 - \lambda)z) \leq \lambda f(w) + (1 - \lambda)f(z)$$

becomes

$$f(x) \leq \frac{z - x}{z - w}f(w) + \frac{x - w}{z - w}f(z)$$

$$\begin{aligned} \text{RHS} &= \frac{z - x}{z - w}f(w) + \frac{x - w}{z - w}f(z) = \frac{z - x}{z - w}f(w) + \left(\frac{x - w}{z - w} + 1 - 1\right)f(z) \\ &= \frac{z - x}{z - w}f(w) + \left(\frac{x - w}{z - w} - 1\right)f(z) + f(z) \\ &= \frac{(z - x)f(w) + (x - w - z + w)f(z)}{z - w} + f(z) \\ &= \frac{(z - x)f(w) + (x - z)f(z)}{z - w} + f(z) \\ &= \frac{-(x - z)f(w) + (x - z)f(z)}{z - w} + f(z) \\ &= \frac{f(z) - f(w)}{z - w}(x - z) + f(z) \end{aligned}$$

$$\begin{aligned} f(x) &\leq \frac{f(z) - f(w)}{z - w} (x - z) + f(z) && \text{subtract } f(z) \\ f(x) - f(z) &\leq \frac{f(z) - f(w)}{z - w} (x - z) && x - z < 0 \\ \frac{f(x) - f(z)}{x - z} &\geq \frac{f(z) - f(w)}{z - w} \end{aligned}$$