

$$\cos 2x = 2 \cos^2 x - 1 \quad \Rightarrow \quad \cos^2 x = \frac{1 + \cos 2x}{2}$$

$$\begin{aligned}\cos^4 x &= (\cos^2 x)^2 = \left(\frac{1 + \cos 2x}{2} \right)^2 = \frac{1}{4} (\cos^2 2x + 2 \cos 2x + 1) \\ &= \frac{1}{4} \left(\frac{1 + \cos 4x}{2} + 2 \cos 2x + 1 \right) \\ &= \frac{1}{8} \cos 4x + \frac{1}{2} \cos 2x + \frac{3}{8}\end{aligned}$$

$$\begin{aligned}\int \cos^4 x \, dx &= \int \frac{1}{8} \cos 4x + \frac{1}{2} \cos 2x + \frac{3}{8} \, dx \\ &= \frac{1}{8} \int \cos 4x \, dx + \frac{1}{2} \int \cos 2x \, dx + \frac{3}{8} \int dx \\ &= \frac{1}{8} \cdot \frac{\sin 4x}{4} + \frac{1}{2} \cdot \frac{\sin 2x}{2} + \frac{3}{8} x + C \\ &= \frac{1}{32} \sin 4x + \frac{1}{4} \sin 2x + \frac{3}{8} x + C\end{aligned}$$