

Sample Problems

1. Consider the straight line $y = \frac{1}{2}x$ on the interval $[0, 6]$. Suppose we rotate the line around the x -axis. Find the volume of the object we obtain by
 - (a) first writing a Riemann sum estimating the volume
 - (b) turning the Riemann sum into an integral and evaluating it.
 - (c) Repeat the previous computations using the function $f(x) = mx$ on the interval $[0, h]$, and r denoting the value of $f(h) = mh$.
2. We are standing on the top of a 60 m tall building. We have to lift a bucket from the ground, using a rope. The bucket weighs 250 N and the rope's unit weight is $2\frac{\text{N}}{\text{m}}$.
 - (a) Find a Riemann sum expressing the amount of work it takes to lift the bucket from the ground to the top of the building.
 - (b) Turn the Riemann sum into an integral and evaluate it to find the work.
3. A cylindrical tank has height 10 m and base radius 3 m. It is full of water, and we have to pump it all out by a pipe that is always leveled at the surface of the water. Assume that 1 m^3 of water weighs 10 000 N, i.e. the 'density' of water is $\delta = 10\,000\frac{\text{N}}{\text{m}^3}$. How much work does it require to pump all water out of the tank?
 - (a) Find a Riemann sum expressing the work.
 - (b) Turn the Riemann sum into an integral and evaluate it.
4. A tank, shaped like a cone has height 10 m and base radius 3 m. It is placed so that the circular part is upward. It is full of water, and we have to pump it all out by a pipe that is always leveled at the surface of the water. Assume that 1 m^3 of water weighs 10 000 N, i.e. the 'density' of water is $\delta = 10\,000\frac{\text{N}}{\text{m}^3}$. How much work does it require to pump all water out of the tank?
 - (a) Find a Riemann sum expressing the work.
 - (b) Turn the Riemann sum into an integral and evaluate it.

Sample Problems - Answers

$$1. \text{ a) } V_{\text{total}} = \sum_{i=0}^{n-1} \frac{\pi x_i^2}{4} \Delta x \quad \text{or} \quad V_{\text{total}} = \sum_{i=1}^n \frac{\pi x_i^2}{4} \Delta x \quad \text{b) } 18\pi \quad \text{c) } \frac{\pi r^2 h}{3}$$

$$2. \text{ a) } W_{\text{total}} = \sum_{i=0}^{n-1} (370 - 2x_i) \Delta x \quad \text{or} \quad W_{\text{total}} = \sum_{i=1}^n (370 - 2x_i) \Delta x \quad \text{b) } 18\,600 \text{ J}$$

$$3. \text{ a) } W_{\text{total}} = \sum_{i=0}^{n-1} (\delta \pi r^2) (10 - x_i) \Delta x \quad \text{or} \quad W_{\text{total}} = \sum_{i=1}^n (\delta \pi r^2) (10 - x_i) \Delta x$$

b) $4500\,000\pi \text{ J} \cong 1.413\,72 \times 10^7 \text{ J}$

$$4. \text{ a) } W_{\text{total}} = \sum_{i=0}^{n-1} \frac{9\delta \pi x_i^2}{100} (10 - x_i) \Delta x \quad \text{or} \quad W_{\text{total}} = \sum_{i=1}^n \frac{9\delta \pi x_i^2}{100} (10 - x_i) \Delta x$$

b) $750\,000 \pi \text{ J} = 2.356\,2 \times 10^6 \text{ J}$

Sample Problems - Solutions

1. Consider the straight line $y = \frac{1}{2}x$ on the interval $[0, 6]$. Suppose we rotate the line around the x -axis. Find the volume of the object we obtain by

a) first writing a Riemann sum estimating the volume

Solution: Let us partition the interval $[0, 6]$ into n equal intervals by $0 = x_0, x_1, x_2, \dots, x_n = 6$, where the points are equally placed, and let Δx denote the length of a subinterval $[x_i, x_{i+1}]$.

We will approximate the function $f(x) = \frac{1}{2}x$ as constant $f(x_i)$ over the entire interval $[x_i, x_{i+1}]$. Then the volume of a slice will be that of a cylinder with radius $f(x_i)$ and height Δx . The volume of the cylindrical slice is

$$V_i = \pi r^2 h = \pi (f(x_i))^2 \Delta x = \pi \left(\frac{x_i}{2}\right)^2 \Delta x = \frac{\pi x_i^2}{4} \Delta x$$

The total volume is

$$V_{\text{total}} = \sum_{i=0}^{n-1} V_i = \sum_{i=0}^{n-1} \frac{\pi x_i^2}{4} \Delta x$$

b) turning the Riemann sum into an integral and evaluating it.

Solution: As n approaches infinity, the Riemann sum approaches the integral

$$\begin{aligned} V_{\text{total}} &= \lim_{n \rightarrow \infty} \sum_{i=0}^{n-1} V_i = \lim_{n \rightarrow \infty} \sum_{i=0}^{n-1} \frac{\pi x_i^2}{4} \Delta x = \int_0^6 \left(\frac{\pi x^2}{4} \right) dx \\ &= \frac{\pi}{4} \int_0^6 x^2 dx = \frac{\pi}{4} \left(\frac{x^3}{3} \right) \Big|_0^6 = \frac{\pi}{4} \left(\frac{6^3}{3} - \frac{0^3}{3} \right) = \frac{\pi}{4} (72) = 18\pi \end{aligned}$$

b) Repeat the previous computations using the function $f(x) = mx$ on the interval $[0, h]$, and r denoting the value of $f(h) = mh$.

$$V_i = \pi r^2 h = \pi (f(x_i))^2 \Delta x = \pi (mx_i)^2 \Delta x = \pi m^2 x_i^2 \Delta x$$

and thus the volume is

$$\begin{aligned} V_{\text{total}} &= \lim_{n \rightarrow \infty} \sum_{i=0}^{n-1} V_i = \lim_{n \rightarrow \infty} \sum_{i=0}^{n-1} \pi m^2 x_i^2 \Delta x = \int_0^h (\pi m^2 x^2) dx = \pi m^2 \int_0^h x^2 dx \\ &= \pi m^2 \left(\frac{x^3}{3} \right) \Big|_0^h = \pi m^2 \left(\frac{h^3}{3} - \frac{0^3}{3} \right) = \frac{\pi m^2 h^3}{3} = \frac{\pi (m^2 h^2) h}{3} = \frac{\pi r^2 h}{3} \end{aligned}$$

2. We are standing on the top of a 60 m tall building. We have to lift a bucket from the ground, using a rope. The bucket weighs 250 N and the rope's unit weight is $2 \frac{\text{N}}{\text{m}}$.
- a) Find a Riemann sum expressing the amount of work it takes to lift the bucket from the ground to the top of the building.

Solution: Let x represent the height of the bucket. Let us partition the interval $[0, 60]$ into n equal intervals by $0 = x_0, x_1, x_2, \dots, x_n = 60$, where the points are equally placed, and let Δx denote the length of a subinterval $[x_i, x_{i+1}]$. We will approximate the height of the bucket to be x_i over the entire interval $[x_i, x_{i+1}]$. Then the force we have to exert while lifting the bucket from height x_i to x_{i+1} is

$$F_i = F_{\text{bucket}} + F_{\text{rope}} = 250 + 2(60 - x_i) = 370 - 2x_i$$

Then the work we have to invest when lifting the bucket from height x_i to x_{i+1} is

$$W_i = F_i \cdot s_i = (370 - 2x_i)(x_{i+1} - x_i) = (370 - 2x_i) \Delta x$$

The total work is then

$$W_{\text{total}} = \sum_{i=0}^{n-1} W_i = \sum_{i=0}^{n-1} F_i \cdot s_i = \sum_{i=0}^{n-1} (370 - 2x_i) \Delta x$$

b) Turn the Riemann sum into an integral and evaluate it to find the work.

Solution: As n approaches infinity, the Riemann sum approaches the integral

$$\begin{aligned} W_{\text{total}} &= \lim_{n \rightarrow \infty} \sum_{i=0}^{n-1} (370 - 2x_i) \Delta x = \int_0^{60} (370 - 2x_i) dx = 370x - x^2 \Big|_0^{60} \\ &= (370(60) - (60)^2) - (370(0) - (0)^2) = 22\,200 - 3600 = 18\,600 \end{aligned}$$

(a) or, the same computation with the units, quite elegant,

$$\begin{aligned} W_{\text{total}} &= \lim_{n \rightarrow \infty} \sum_{i=0}^{n-1} \left(370 \text{ N} - 2 \frac{\text{N}}{\text{m}} x_i \right) \Delta x = \int_0^{60 \text{ m}} \left(370 \text{ N} - 2 \frac{\text{N}}{\text{m}} x \right) dx \\ &= \left(370 \text{ N} x - x^2 \frac{\text{N}}{\text{m}} \right) \Big|_{0 \text{ m}}^{60 \text{ m}} \\ &= \left[370 \text{ N} (60 \text{ m}) - (60 \text{ m})^2 \frac{\text{N}}{\text{m}} \right] - \left[370 \text{ N} (0 \text{ m}) - (0 \text{ m})^2 \frac{\text{N}}{\text{m}} \right] \\ &= 22\,200 \text{ N m} - 3600 \text{ N m} = 18\,600 \text{ N m} \\ &= 18\,600 \text{ J} \end{aligned}$$

3. A cylindrical tank has height 10 m and base radius 3 m. It is full of water, and we have to pump it all out by a pipe that is always leveled at the surface of the water. Assume that 1 m^3 of water weighs 10 000 N, i.e. the 'density' of water is $\delta = 10\,000 \frac{\text{N}}{\text{m}^3}$. How much work does it require to pump all water out of the tank?

a) Find a Riemann sum expressing the work.

Solution: Let x represent the water level. Let us partition the interval $[0, 10]$ into n equal intervals by $0 = x_0, x_1, x_2, \dots, x_n = 10$, where the points are equally placed, and let Δx denote the length of a subinterval $[x_i, x_{i+1}]$. We will approximate the water level to be x_i over the

entire interval $[x_i, x_{i+1}]$. Then the force we have to exert when pumping out the water between heights x_i and x_{i+1} is the weight of a cylindrical 'slice' with volume

$$\begin{aligned} V_i &= \pi r^2 \Delta x \quad \text{the weight is then} \quad F_i = \delta V_i \\ F_i &= \delta \pi r^2 \Delta x \end{aligned}$$

The water needs to be lifted to the top of the tank, and so

$$s_i = 10 - x_i$$

Then the work we have to invest when lifting the slice of water from height x_i to 10 is

$$W_i = F_i \cdot s_i = (\delta \pi r^2 \Delta x) (10 - x_i) = (\delta \pi r^2) (10 - x_i) \Delta x$$

The total work is then is

$$W_{\text{total}} = \sum_{i=0}^{n-1} W_i = \sum_{i=0}^{n-1} F_i \cdot s_i = \sum_{i=0}^{n-1} (\delta \pi r^2) (10 - x_i) \Delta x$$

b) Turn the Riemann sum into an integral and evaluate it.

Solution: As n approaches infinity, the Riemann sum approaches the integral

$$\begin{aligned} W_{\text{total}} &= \lim_{n \rightarrow \infty} \sum_{i=0}^{n-1} (\delta \pi r^2) (10 - x_i) \Delta x = \int_0^{10} (\delta \pi r^2) (10 - x) dx = \delta \pi r^2 \int_0^{10} (10 - x) dx \\ &= 10\,000\pi (9) \int_0^{10} (10 - x) dx = 90\,000\pi \left(10x - \frac{x^2}{2} \right) \Big|_0^{10} \\ &= 90\,000\pi \left[\left(10(10) - \frac{(10)^2}{2} \right) - \left(10(0) - \frac{(0)^2}{2} \right) \right] \\ &= 90\,000\pi (50) = 4\,500\,000\pi \cong 1.413\,72 \times 10^7 \text{ J} \end{aligned}$$

4. A tank, shaped like a cone has height 10 m and base radius 3 m. It is placed so that the circular part is upward. It is full of water, and we have to pump it all out by a pipe that is always leveled at the surface of the water. Assume that 1 m^3 of water weighs 10 000 N, i.e. the 'density' of water is $\delta = 10\,000 \frac{\text{N}}{\text{m}^3}$. How much work does it require to pump all water out of the tank?

a) Find a Riemann sum expressing the work.

Solution: Let x represent the water level. Let us partition the interval $[0, 10]$ into n equal intervals by $0 = x_0, x_1, x_2, \dots, x_n = 10$, where the points are equally placed, and let Δx denote the length of a subinterval $[x_i, x_{i+1}]$. We will approximate the water level to be x_i over the entire interval $[x_i, x_{i+1}]$. Then the force we have to exert when pumping out the water between heights x_i and x_{i+1} is the weight of a cylindrical 'slice' with volume $\pi r_i^2 h$ where $h = \Delta x$ and

r_i is the radius at height x_i . Clearly $\tan \alpha = \frac{r_i}{x_i} = \frac{3}{10}$ which gives us that $r_i = \frac{3}{10}x_i$

$$\begin{aligned} V_i &= \pi r_i^2 \Delta x = \pi \left(\frac{3}{10} x_i \right)^2 \Delta x = \frac{9\pi x_i^2}{100} \Delta x \text{ the weight is then } F_i = \delta V_i \\ F_i &= \frac{9\delta\pi x_i^2}{100} \Delta x \end{aligned}$$

The water needs to be lifted to the top of the tank, and so

$$s_i = 10 - x_i$$

Then the work we have to invest when lifting the slice of water from height x_i to 10 is

$$W_i = F_i \cdot s_i = \left(\frac{9\delta\pi x_i^2}{100} \Delta x \right) (10 - x_i) = \frac{9\delta\pi x_i^2}{100} (10 - x_i) \Delta x$$

The total work is then is

$$W_{\text{total}} = \sum_{i=0}^{n-1} W_i = \sum_{i=0}^{n-1} F_i \cdot s_i = \sum_{i=0}^{n-1} \frac{9\delta\pi x_i^2}{100} (10 - x_i) \Delta x$$

b) Turn the Riemann sum into an integral and evaluate it.

Solution: As n approaches infinity, the Riemann sum approaches the integral

$$\begin{aligned} W_{\text{total}} &= \lim_{n \rightarrow \infty} \sum_{i=0}^{n-1} \frac{9\delta\pi x_i^2}{100} (10 - x_i) \Delta x = \int_0^{10} \frac{9\delta\pi x^2}{100} (10 - x) dx \\ &= \frac{9\delta\pi}{100} \int_0^{10} x^2 (10 - x) dx = \frac{9(10000)\pi}{100} \int_0^{10} x^2 (10 - x) dx \\ &= 900\pi \int_0^{10} (10x^2 - x^3) dx \\ &= 900\pi \left(\frac{10x^3}{3} - \frac{x^4}{4} \right) \Big|_0^{10} \\ &= 900\pi \left[\left(\frac{10(10)^3}{3} - \frac{10^4}{4} \right) - \left(\frac{0^3}{3} - \frac{0^4}{4} \right) \right] \\ &= 900\pi \left(\frac{10^4}{3} - \frac{10^4}{4} \right) = 900\pi \left(\frac{1}{12} \right) 10^4 \\ &= 750\,000\pi \end{aligned}$$