

Definition: $\lim_{n \rightarrow \infty} \left(1 + \frac{1}{n}\right)^n = e$

Theorem: $\lim_{n \rightarrow \infty} \left(1 + \frac{a}{n}\right)^n = e^a$

Proof: If $a = 0$, then clearly $\lim_{n \rightarrow \infty} \left(1 + \frac{a}{n}\right)^n = \lim_{n \rightarrow \infty} \left(1 + \frac{0}{n}\right)^n = \lim_{n \rightarrow \infty} 1^n = 1$. If $a \neq 0$, then

$$\lim_{n \rightarrow \infty} \left(1 + \frac{a}{n}\right)^n = \lim_{n \rightarrow \infty} \left(1 + \frac{1}{\frac{n}{a}}\right)^n = \lim_{n \rightarrow \infty} \left(1 + \frac{1}{\frac{n}{a}}\right)^{\frac{n}{a} \cdot a} = \lim_{n \rightarrow \infty} \left[\left(1 + \frac{1}{\frac{n}{a}}\right)^{\frac{n}{a}} \right]^a$$

Define $m = \frac{n}{a}$. Since a is fixed, m approaches infinity (or negative infinity if a is negative) as n approaches infinity.

$$\left[\lim_{n \rightarrow \infty} \left(1 + \frac{1}{\frac{n}{a}}\right)^{\frac{n}{a}} \right]^a = \left[\lim_{m \rightarrow \infty} \left(1 + \frac{1}{m}\right)^m \right]^a = e^a$$

Practice Problems

Assume that

$$\lim_{n \rightarrow \infty} \left(1 + \frac{1}{n}\right)^n = e \quad \text{and} \quad \lim_{n \rightarrow -\infty} \left(1 + \frac{1}{n}\right)^n = e$$

and compute each of the following limits.

1. $\lim_{n \rightarrow \infty} \left(1 + \frac{1}{n}\right)^{n+2}$

3. $\lim_{n \rightarrow \infty} \left(1 + \frac{5}{n}\right)^n$

5. $\lim_{n \rightarrow \infty} \left(\frac{n}{n+2}\right)^n$

2. $\lim_{n \rightarrow \infty} \left(1 + \frac{1}{n}\right)^{2n}$

4. $\lim_{n \rightarrow \infty} \left(1 - \frac{1}{n}\right)^n$

Answers - Practice Problems

$$1.) \ e \quad 2.) \ e^2 \quad 3.) \ e^5 \quad 4.) \ \frac{1}{e} \quad 5.) \ \frac{1}{e^2}$$

Solutions - Practice Problems

Assume that

$$\lim_{n \rightarrow \infty} \left(1 + \frac{1}{n}\right)^n = e \quad \text{and} \quad \lim_{n \rightarrow -\infty} \left(1 + \frac{1}{n}\right)^n = e$$

and compute each of the following limits.

$$1. \ \lim_{n \rightarrow \infty} \left(1 + \frac{1}{n}\right)^{n+2}$$

$$\lim_{n \rightarrow \infty} \left(1 + \frac{1}{n}\right)^{n+2} = \lim_{n \rightarrow \infty} \left(1 + \frac{1}{n}\right)^n \left(1 + \frac{1}{n}\right)^2 = \lim_{n \rightarrow \infty} \left(1 + \frac{1}{n}\right)^n \lim_{n \rightarrow \infty} \left(1 + \frac{1}{n}\right)^2 = e \cdot 1 = e$$

$$2. \ \lim_{n \rightarrow \infty} \left(1 + \frac{1}{n}\right)^{2n}$$

$$\lim_{n \rightarrow \infty} \left(1 + \frac{1}{n}\right)^{2n} = \lim_{n \rightarrow \infty} \left[\left(1 + \frac{1}{n}\right)^n\right]^2 = \left[\lim_{n \rightarrow \infty} \left(1 + \frac{1}{n}\right)^n\right]^2 = e^2$$

$$3. \ \lim_{n \rightarrow \infty} \left(1 + \frac{5}{n}\right)^n$$

$$\lim_{n \rightarrow \infty} \left(1 + \frac{5}{n}\right)^n = \lim_{n \rightarrow \infty} \left(1 + \frac{1}{\frac{n}{5}}\right)^n = \lim_{n \rightarrow \infty} \left(1 + \frac{1}{\frac{n}{5}}\right)^{\frac{n}{5} \cdot 5} = \lim_{n \rightarrow \infty} \left[\left(1 + \frac{1}{\frac{n}{5}}\right)^{\frac{n}{5}}\right]^5$$

Let $m = \frac{n}{5}$. As n approaches infinity, so does m .

$$\left[\lim_{n \rightarrow \infty} \left(1 + \frac{1}{\frac{n}{5}}\right)^{\frac{n}{5}}\right]^5 = \left[\lim_{m \rightarrow \infty} \left(1 + \frac{1}{m}\right)^m\right]^5 = e^5$$

4. $\lim_{n \rightarrow \infty} \left(1 - \frac{1}{n}\right)^n$

$$\begin{aligned} \lim_{n \rightarrow \infty} \left(1 - \frac{1}{n}\right)^n &= \lim_{n \rightarrow \infty} \left(1 + \frac{-1}{n}\right)^n = \lim_{n \rightarrow \infty} \left(1 + \frac{1}{\frac{n}{-1}}\right)^n = \lim_{n \rightarrow \infty} \left(1 + \frac{1}{\frac{n}{-1}}\right)^{\frac{n}{-1} \cdot (-1)} \\ &= \lim_{n \rightarrow \infty} \left[\left(1 + \frac{1}{-n}\right)^{(-n)} \right]^{(-1)} = \left[\lim_{n \rightarrow \infty} \left(1 + \frac{1}{-n}\right)^{(-n)} \right]^{(-1)} \end{aligned}$$

Let $m = -n$

$$\left[\lim_{n \rightarrow \infty} \left(1 + \frac{1}{-n}\right)^{(-n)} \right]^{(-1)} = \left[\lim_{m \rightarrow -\infty} \left(1 + \frac{1}{m}\right)^m \right]^{(-1)} = e^{-1} = \frac{1}{e}$$

5. $\lim_{n \rightarrow \infty} \left(\frac{n}{n+2}\right)^n$

$$\begin{aligned} \lim_{n \rightarrow \infty} \left(\frac{n}{n+2}\right)^n &= \lim_{n \rightarrow \infty} \left(\frac{1}{\frac{n+2}{n}}\right)^n = \lim_{n \rightarrow \infty} \left(\left(\frac{n+2}{n}\right)^{(-1)}\right)^n = \lim_{n \rightarrow \infty} \left[\left(1 + \frac{2}{n}\right)^n\right]^{(-1)} \\ &= \left[\lim_{n \rightarrow \infty} \left(1 + \frac{2}{n}\right)^n \right]^{(-1)} = (e^2)^{(-1)} = e^{-2} = \frac{1}{e^2} \end{aligned}$$

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