

Definition: Given a sequence of numbers  $\{a_n\}$ , the expression

$$a_1 + a_2 + a_3 + \dots + a_n + \dots$$

is called an **infinite series**. The number  $a_n$  is the ***n*th term** of the series. The sequence  $\{s_n\}$  defined by

$$\begin{aligned} s_1 &= a_1 \\ s_2 &= a_1 + a_2 \\ s_3 &= a_1 + a_2 + a_3 \\ &\vdots \\ s_n &= a_1 + a_2 + a_3 + \dots + a_n = \sum_{k=1}^n a_k \end{aligned}$$

is called the **sequence of partial sums** of the series, the number  $s_n$  being the ***n*th partial sum**. If the sequence of partial sums converges to a limit  $L$ , we say that the series converges and that its **sum** is  $L$ . In this case, we also write

$$a_1 + a_2 + a_3 + \dots + a_n + \dots = \sum_{n=1}^{\infty} a_n = L$$

If the sequence of partial sums of the series does not converge, we say that the series **diverges**.

Definition: **Geometric series** are of the form

$$a + ar + ar^2 + \dots + ar^n + \dots = \sum_{n=0}^{\infty} ar^n$$

where  $a$  and  $r$  are fixed real numbers and  $a \neq 0$ .

Theorem: If  $r = 1$ , then  $s_n = na$  and if  $r \neq 1$ , then  $s_n = a \frac{1 - r^n}{1 - r}$ .

proof. Case 1. If  $r = 1$ , then  $a_n = a + ar + ar^2 + \dots + ar^{n-1} = a + a + \dots + a = na$ .

Case 2. If  $r \neq 1$ , then

$$\begin{aligned} a + ar + ar^2 + \dots + ar^{n-1} &= s_n && \text{multiply both sides by } r \\ ar + ar^2 + ar^3 + \dots + ar^n &= rs_n && \text{subtract} \\ \Downarrow &&& \\ a - ar^n &= s_n - rs_n \\ a(1 - r^n) &= s_n(1 - r) && \text{divide by } 1 - r \\ a \frac{1 - r^n}{1 - r} &= s_n \end{aligned}$$

Theorem: If  $|r| < 1$ , the geometric series  $a + ar + ar^2 + \dots + ar^{n-1} + \dots$  converges to  $\frac{a}{1-r}$  and if  $|r| \geq 1$ , the series diverges.

proof: If  $r = 1$ , then the sequence is constant. In this case, if  $a = 0$ , the sequence is the constant zero sequence. Then the series converges to zero. If  $a \neq 0$ , the series diverges to infinity or negative infinity, depending on the sign of  $a$ .

Suppose now that  $r \neq 1$ . The infinite sum is defined as the limit of the partial sums:

$$s = \lim_{n \rightarrow \infty} s_n = \lim_{n \rightarrow \infty} a \frac{1 - r^n}{1 - r} = a \frac{1 - \lim_{n \rightarrow \infty} r^n}{1 - r}$$

Now if  $|r| > 1$ , then  $r^n$  diverges and so there is no infinite sum defined. If  $|r| < 1$ , then  $\lim_{n \rightarrow \infty} r^n = 0$  and so

$$s = \lim_{n \rightarrow \infty} s_n = \lim_{n \rightarrow \infty} a \frac{1 - r^n}{1 - r} = a \frac{1 - \lim_{n \rightarrow \infty} r^n}{1 - r} = a \frac{1}{1 - r} = \frac{a}{1 - r}$$

## Sample Problems

In each case, compute the sum of the infinite series given.

1.  $\sum_{n=0}^{\infty} \frac{2}{3^n}$
2.  $\sum_{n=0}^{\infty} \left(\frac{2}{3}\right)^{n+1}$
3.  $\sum_{n=0}^{\infty} \frac{2^{2n+1}}{3^n}$
4.  $\sum_{n=0}^{\infty} (-1)^n \frac{5^n}{3^{2n-1}}$

## Practice Problems

In each case, determine whether the given geometric series converges or diverges. If converges, find its sum.

1.  $\sum_{n=1}^{\infty} \left(\frac{1}{10}\right)^n$
5.  $\sum_{n=0}^{\infty} \frac{2^{n+1}(-3)^{n+1}}{5^{n-2}}$
9.  $\sum_{n=2}^{\infty} \frac{2^n 3^{n+1}}{7^{n-1}}$
2.  $\sum_{n=0}^{\infty} (-1)^n \frac{2^{n+1}}{3^{n-1}}$
6.  $\sum_{n=0}^{\infty} \frac{9^n}{10^{n+1}}$
10.  $\sum_{n=0}^{\infty} \frac{1}{e^n}$
3.  $\sum_{n=0}^{\infty} (-1)^n \frac{2^{2n}}{3^{n+1}}$
7.  $\sum_{n=1}^{\infty} \frac{3}{10^n}$
4.  $\sum_{n=0}^{\infty} \frac{2^{2n+1}}{5^{n-2}}$
8.  $\sum_{n=1}^{\infty} \frac{3}{(-10)^n}$
- 11\*.  $\sum_{n=1}^{\infty} \frac{2n-1}{3^{n+1}} = \frac{1}{9} + \frac{3}{27} + \frac{5}{81} + \frac{7}{243} + \dots$

## Answers - Sample Problems

- 1.) 3      2.) 2      3.) diverges      4.)  $\frac{27}{14}$

## Answers - Practice Problems

- 1.)  $\frac{1}{9}$       2.)  $\frac{18}{5}$       3.) diverges      4.) 250      5.) diverges      6.) 1      7.)  $\frac{1}{3}$       8.)  $-\frac{3}{11}$       9.) 108
- 10.)  $\frac{e}{e-1}$       11.)  $\frac{1}{3}$

## Sample Problems - Solutions

In each case, compute the sum of the infinite series given.

1.  $\sum_{n=0}^{\infty} \frac{2}{3^n}$

This sequence is  $2, \frac{2}{3}, \frac{2}{9}, \frac{2}{27}, \dots$ . Thus  $a = 2$  and  $r = \frac{1}{3}$ . The sum of the series exists and can be computed as

$$s = \frac{a}{1-r} = \frac{2}{1-\frac{1}{3}} = \frac{2}{\frac{2}{3}} = 3$$

2.  $\sum_{n=0}^{\infty} \left(\frac{2}{3}\right)^{n+1}$

This sequence is  $\frac{2}{3}, \frac{4}{9}, \frac{8}{27}, \dots$ . Thus  $a = \frac{2}{3}$  and  $r = \frac{2}{3}$ . The sum of the series exists and can be computed as

$$s = \frac{a}{1-r} = \frac{\frac{2}{3}}{1-\frac{2}{3}} = \frac{\frac{2}{3}}{\frac{1}{3}} = 2$$

3.  $\sum_{n=0}^{\infty} \frac{2^{2n+1}}{3^n}$

Sometimes a bit of algebra helps more than writing out the first few terms immediately. We can re-write  $2^{2n+1}$  as

$$2^{2n+1} = 2^{2n} \cdot 2 = (2^2)^n \cdot 2 = 4^n \cdot 2$$

and so

$$\sum_{n=0}^{\infty} \frac{2^{2n+1}}{3^n} = \sum_{n=0}^{\infty} \frac{4^n \cdot 2}{3^n} = \sum_{n=0}^{\infty} 2 \left(\frac{4}{3}\right)^n$$

This sequence is  $2, \frac{8}{3}, \frac{32}{9}, \dots$ . Thus  $a = 2$  and  $r = \frac{4}{3}$ . Since  $\frac{4}{3} > 1$ , this series diverges.

$$4. \sum_{n=0}^{\infty} (-1)^n \frac{5^n}{3^{2n-1}}$$

$$\sum_{n=0}^{\infty} (-1)^n \frac{5^n}{3^{2n-1}} = \sum_{n=0}^{\infty} (-1)^n \frac{5^n}{\frac{3^{2n}}{3}} = \sum_{n=0}^{\infty} (-1)^n \frac{3 \cdot 5^n}{(3^2)^n} = \sum_{n=0}^{\infty} (-1)^n \frac{3 \cdot 5^n}{9^n} = \sum_{n=0}^{\infty} 3 \cdot \left(-\frac{5}{9}\right)^n$$

Thus  $a = 3$  and  $r = -\frac{5}{9}$ . Then the sum of the series exists and can be computed as

$$s = \frac{a}{1-r} = \frac{3}{1 - \left(-\frac{5}{9}\right)} = \frac{3}{\frac{14}{9}} = \frac{27}{14}$$