

Definition: We define the hyperbolic functions as

$$\begin{aligned}\sinh x &= \frac{1}{2} (e^x - e^{-x}) & \cosh x &= \frac{1}{2} (e^x + e^{-x}) & \tanh x &= \frac{\sinh x}{\cosh x} = \frac{e^x - e^{-x}}{e^x + e^{-x}} \\ \operatorname{csch} x &= \frac{1}{\sinh x} = \frac{2}{e^x - e^{-x}} & \operatorname{sech} x &= \frac{1}{\cosh x} = \frac{2}{e^x + e^{-x}} & \coth x &= \frac{\cosh x}{\sinh x} = \frac{e^x + e^{-x}}{e^x - e^{-x}}\end{aligned}$$

Theorem: $\sinh 0 = 0$ and $\cosh 0 = 1$

Proof:

$$\sinh 0 = \frac{1}{2} (e^0 - e^{-0}) = \frac{1}{2} (1 - 1) = 0 \quad \text{and} \quad \cosh 0 = \frac{1}{2} (e^0 + e^{-0}) = \frac{1}{2} (1 + 1) = 1$$

Theorem: $\sinh x$ and $\tanh x$ are odd, and $\cosh x$ is even.

Proof: Let x be any real number.

$$\begin{aligned}\sinh(-x) &= \frac{1}{2} (e^{-x} - e^{-(-x)}) = \frac{1}{2} (e^{-x} - e^x) = -\frac{1}{2} (e^x - e^{-x}) = -\sinh x \\ \cosh(-x) &= \frac{1}{2} (e^{-x} + e^{-(-x)}) = \frac{1}{2} (e^{-x} + e^x) = \frac{1}{2} (e^x + e^{-x}) = \cosh x\end{aligned}$$

$$\tanh(-x) = \frac{\sinh(-x)}{\cosh(-x)} = \frac{-\sinh x}{\cosh x} = -\tanh x$$

Limits

Theorem: $\lim_{x \rightarrow \infty} \sinh x = \infty$ and $\lim_{x \rightarrow -\infty} \sinh x = -\infty$

Proof: $\lim_{x \rightarrow \infty} \sinh x = \lim_{x \rightarrow \infty} \frac{1}{2} \begin{pmatrix} e^x - e^{-x} \\ \downarrow \quad \downarrow \\ \infty \quad 0 \end{pmatrix} = \infty$ Since $\sinh x$ is odd, $\lim_{x \rightarrow -\infty} \sinh x = -\infty$ follows.

Theorem: $\lim_{x \rightarrow \infty} \cosh x = \lim_{x \rightarrow -\infty} \cosh x = \infty$

Proof: $\lim_{x \rightarrow \infty} \cosh x = \lim_{x \rightarrow \infty} \frac{1}{2} \begin{pmatrix} e^x + e^{-x} \\ \downarrow \quad \downarrow \\ \infty \quad 0 \end{pmatrix} = \infty$ Since $\cosh x$ is even, $\lim_{x \rightarrow -\infty} \cosh x = \infty$ follows.

Theorem: $\lim_{x \rightarrow \infty} \tanh x = 1$ and $\lim_{x \rightarrow -\infty} \tanh x = -1$

Proof:

$$\lim_{x \rightarrow \infty} \tanh x = \lim_{x \rightarrow \infty} \frac{e^x - e^{-x}}{e^x + e^{-x}} = \lim_{x \rightarrow \infty} \frac{e^x - \frac{1}{e^x}}{e^x + \frac{1}{e^x}} = \lim_{x \rightarrow \infty} \frac{e^x \left(1 - \frac{1}{e^{2x}}\right)}{e^x \left(1 + \frac{1}{e^{2x}}\right)} = \lim_{x \rightarrow \infty} \frac{1 - \frac{1}{e^{2x}}}{1 + \frac{1}{e^{2x}}} = 1$$

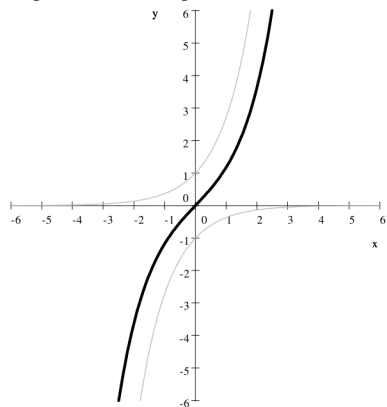
Since $\tanh x$ is odd, $\lim_{x \rightarrow -\infty} \tanh x = -1$ follows.

Graphs

$$y = \sinh x = \frac{1}{2}(e^x - e^{-x})$$

domain: $\mathbb{R}$ range: $\mathbb{R}$

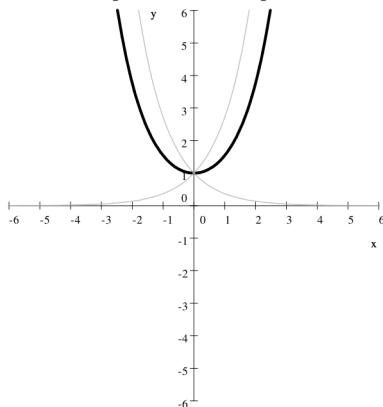
The grey graphs are
 $y = e^x$ and $y = -e^{-x}$.



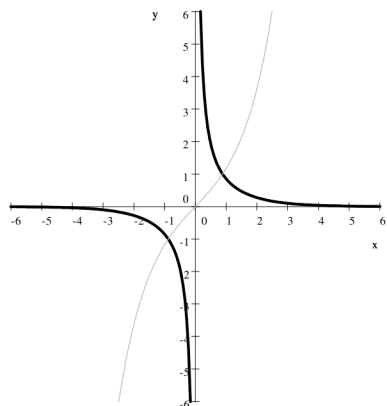
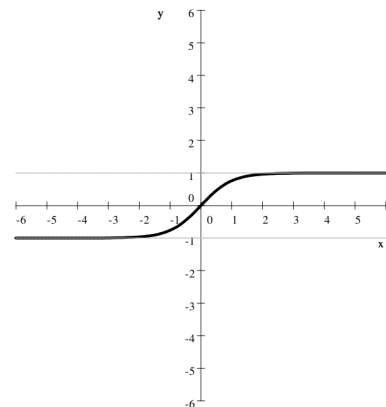
$$y = \cosh x = \frac{1}{2}(e^x + e^{-x})$$

domain: $\mathbb{R}$ range: $[1, \infty)$

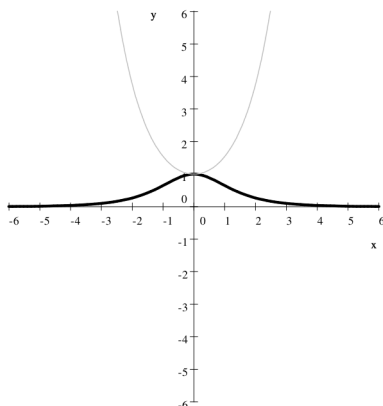
The grey graphs are
 $y = e^x$ and $y = e^{-x}$.



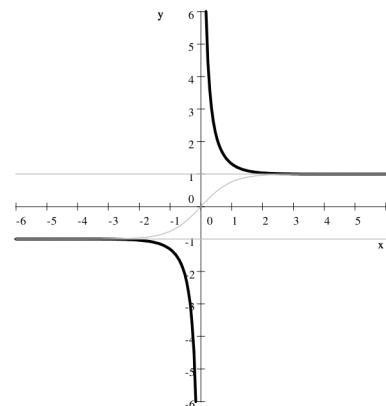
$$y = \tanh x = \frac{e^x - e^{-x}}{e^x + e^{-x}}$$

domain: $\mathbb{R}$ range: $(-1, 1)$ 

$$y = \operatorname{csch} x = \frac{1}{\sinh x} = \frac{2}{e^x - e^{-x}}$$

domain: $(-\infty, 0) \cup (0, \infty)$ range: $(-\infty, 0) \cup (0, \infty)$ 

$$y = \operatorname{sech} x = \frac{1}{\cosh x} = \frac{2}{e^x + e^{-x}}$$

domain: $\mathbb{R}$ range: $(0, 1]$ 

$$y = \operatorname{coth} x = \frac{e^x + e^{-x}}{e^x - e^{-x}}$$

domain: $(-\infty, 0) \cup (0, \infty)$ range: $(-\infty, -1) \cup (1, \infty)$

Identities

Theorem: $\boxed{\cosh^2 x - \sinh^2 x = 1}$ for all real numbers x . Also, $\boxed{\tanh^2 x = 1 - \operatorname{sech}^2 x \text{ and } \coth^2 x = 1 + \operatorname{csch}^2 x}$

Proof: Let x be any real number.

$$\cosh^2 x - \sinh^2 x = \left(\frac{1}{2}(e^x + e^{-x})\right)^2 - \left(\frac{1}{2}(e^x - e^{-x})\right)^2 = \frac{1}{4}(e^{2x} + e^{-2x} + 2) - \frac{1}{4}(e^{2x} + e^{-2x} - 2) = \frac{1}{4}(4) = 1$$

We can derive additional identities from $\cosh^2 x - \sinh^2 x = 1$ by dividing by $\cosh^2 x$ or by $\sinh^2 x$:

$$\begin{array}{ll} \cosh^2 x - \sinh^2 x &= 1 \\ \frac{\cosh^2 x}{\cosh^2 x} - \frac{\sinh^2 x}{\cosh^2 x} &= \frac{1}{\cosh^2 x} \\ 1 - \tanh^2 x &= \operatorname{sech}^2 x \\ 1 - \operatorname{sech}^2 x &= \tanh^2 x \end{array} \qquad \begin{array}{ll} \cosh^2 x - \sinh^2 x &= 1 \\ \frac{\cosh^2 x}{\sinh^2 x} - \frac{\sinh^2 x}{\sinh^2 x} &= \frac{1}{\sinh^2 x} \\ \coth^2 x - 1 &= \operatorname{csch}^2 x \\ \coth^2 x &= 1 + \operatorname{csch}^2 x \end{array}$$

Theorem: ("double-angle" formulas) $\boxed{\sinh 2x = 2 \sinh x \cosh x \text{ and } \cosh 2x = \cosh^2 x + \sinh^2 x}$ for all real numbers x .

Proof: Let x be any real number.

$$\begin{aligned} 2 \sinh x \cosh x &= 2 \cdot \frac{1}{2}(e^x - e^{-x}) \frac{1}{2}(e^x + e^{-x}) = \frac{1}{2}(e^{2x} - e^{-2x}) = \sinh 2x \\ \cosh^2 x + \sinh^2 x &= \left(\frac{1}{2}(e^x + e^{-x})\right)^2 + \left(\frac{1}{2}(e^x - e^{-x})\right)^2 = \frac{1}{4}(e^{2x} + e^{-2x} + 2) + \frac{1}{4}(e^{2x} + e^{-2x} - 2) \\ &= \frac{1}{4}(2e^{2x} + 2e^{-2x}) = \frac{1}{2}(e^{2x} + e^{-2x}) = \cosh 2x \end{aligned}$$

The double-angle formula for $\cosh x$ also has three forms. Recall that $\cosh^2 x - \sinh^2 x = 1$

$$\cosh 2x = \cosh^2 x + \sinh^2 x = 1 + 2 \sinh^2 x = 2 \cosh^2 x - 1$$

From this, we can solve for $\sinh^2 x$ and $\cosh^2 x$:

$$\boxed{\cosh^2 x = \frac{1}{2}(\cosh 2x + 1) \text{ and } \sinh^2 x = \frac{1}{2}(\cosh 2x - 1)}$$

Derivatives

Theorem: $\boxed{\frac{d}{dx} \sinh x = \cosh x \quad \text{and} \quad \frac{d}{dx} \cosh x = \sinh x}$

Proof:

$$\frac{d}{dx} \sinh x = \frac{d}{dx} \left(\frac{1}{2} (e^x - e^{-x}) \right) = \frac{1}{2} (e^x - (-e^{-x})) = \frac{1}{2} (e^x + e^{-x}) = \cosh x \quad \text{and}$$

$$\frac{d}{dx} \cosh x = \frac{d}{dx} \left(\frac{1}{2} (e^x + e^{-x}) \right) = \frac{1}{2} (e^x + (-e^{-x})) = \frac{1}{2} (e^x - e^{-x}) = \sinh x$$

Theorem: $\boxed{\frac{d}{dx} \tanh x = \operatorname{sech}^2 x \quad \text{and} \quad \frac{d}{dx} \coth x = -\operatorname{csch}^2 x}$

Proof:

$$\begin{aligned} \frac{d}{dx} \tanh x &= \frac{d}{dx} \left(\frac{\sinh x}{\cosh x} \right) = \frac{\left(\frac{d}{dx} \sinh x \right) \cosh x - \sinh x \left(\frac{d}{dx} \cosh x \right)}{\cosh^2 x} = \frac{\cosh x \cdot \cosh x - \sinh x \cdot \sinh x}{\cosh^2 x} \\ &= \frac{\cosh^2 x - \sinh^2 x}{\cosh^2 x} = \frac{1}{\cosh^2 x} = \operatorname{sech}^2 x \end{aligned}$$

$$\begin{aligned} \frac{d}{dx} \coth x &= \frac{d}{dx} \left(\frac{\cosh x}{\sinh x} \right) = \frac{\left(\frac{d}{dx} \cosh x \right) \sinh x - \cosh x \left(\frac{d}{dx} \sinh x \right)}{\sinh^2 x} = \frac{\sinh x \cdot \sinh x - \cosh x \cdot \cosh x}{\sinh^2 x} \\ &= \frac{\sinh^2 x - \cosh^2 x}{\sinh^2 x} = \frac{-(\cosh^2 x - \sinh^2 x)}{\sinh^2 x} = \frac{-1}{\sinh^2 x} = -\operatorname{csch}^2 x \end{aligned}$$

Theorem: $\boxed{\frac{d}{dx} \operatorname{sech} x = -\operatorname{sech} x \tanh x \quad \text{and} \quad \frac{d}{dx} \operatorname{csch} x = -\operatorname{csch} x \coth x}$

Proof:

$$\frac{d}{dx} \operatorname{sech} x = \frac{d}{dx} (\cosh x)^{-1} = (-1) (\cosh x)^{-2} \left(\frac{d}{dx} \cosh x \right) = -\frac{1}{\cosh^2 x} (\sinh x) = -\frac{1}{\cosh x} \cdot \frac{\sinh x}{\cosh x} = -\operatorname{sech} x \tanh x$$

$$\frac{d}{dx} \operatorname{csch} x = \frac{d}{dx} (\sinh x)^{-1} = (-1) (\sinh x)^{-2} \left(\frac{d}{dx} \sinh x \right) = -\frac{1}{\sinh^2 x} (\cosh x) = -\frac{1}{\sinh x} \cdot \frac{\cosh x}{\sinh x} = -\operatorname{csch} x \coth x$$

Antiderivatives

Theorem: $\int \sinh x \, dx = \cosh x + C$ and $\int \cosh x \, dx = \sinh x + C$

Proof: These follow from the differentiation formulas.

Theorem: $\int \tanh x \, dx = \ln(\cosh x) + C$ and $\int \coth x \, dx = \ln|\sinh x| + C$

Proof: We compute $\int \tanh x \, dx$ first. Let $u = \cosh x$. Then $du = \sinh x \, dx$ and so

$$\int \tanh x \, dx = \int \frac{\sinh x}{\cosh x} \, dx = \int \frac{1}{\cosh x} (\sinh x \, dx) = \int \frac{1}{u} \, du = \ln|u| + C = \ln|\cosh x| + C$$

Since $\cosh x$ is always positive, we do not need the absolute value sign.

$$\int \tanh x \, dx = \ln(\cosh x) + C$$

Now for $\int \coth x \, dx$: Let $u = \sinh x$. Then $du = \cosh x \, dx$ and so

$$\int \coth x \, dx = \int \frac{\cosh x}{\sinh x} \, dx = \int \frac{1}{\sinh x} (\cosh x \, dx) = \int \frac{1}{u} \, du = \ln|u| + C = \ln|\sinh x| + C$$

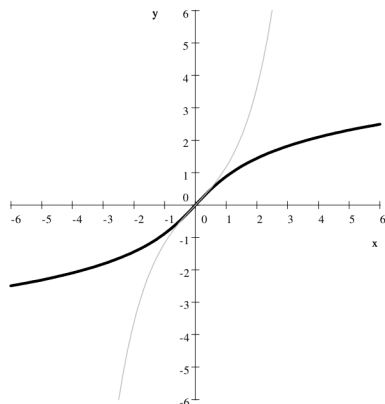
Inverse Hyperbolic Functions

We can easily define $\sinh^{-1} x$ because $\sinh x$ is one-to-one. For $\cosh^{-1} x$, we will need to restrict the domain of $\cosh x$ to $[0, \infty)$. $\tanh x$ is also one-to-one, so $\tanh^{-1} x$ is easily defined.

$$y = \sinh^{-1} x$$

domain: $\mathbb{R}$

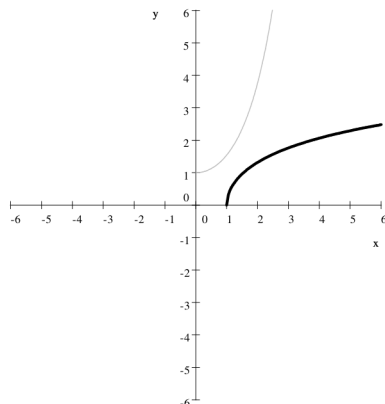
range: $\mathbb{R}$



$$y = \cosh^{-1} x$$

domain: $[1, \infty)$

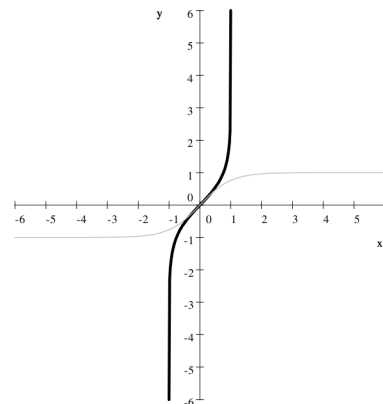
range: $[0, \infty)$



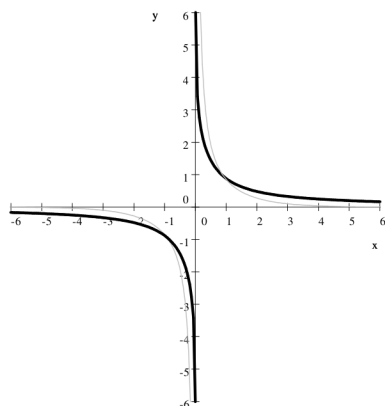
$$y = \tanh^{-1} x$$

domain: $(-1, 1)$

range: $\mathbb{R}$



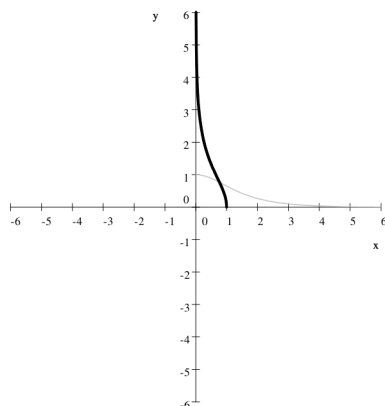
Similarly, the domain of $\operatorname{sech} x$ needs to be restricted to $[0, \infty)$. Since $\operatorname{csch} x$ and $\operatorname{coth} x$ are one-to-one, the inverse functions automatically exist.



$$y = \operatorname{csch}^{-1} x$$

domain: $(-\infty, 0) \cup (0, \infty)$

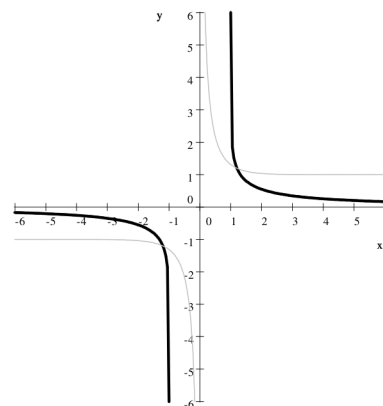
range: $(-\infty, 0) \cup (0, \infty)$



$$y = \operatorname{sech}^{-1} x$$

domain: $(0, 1]$

range: $[0, \infty)$



$$y = \operatorname{coth}^{-1} x$$

domain: $(-\infty, -1) \cup (1, \infty)$

range: $(-\infty, 0) \cup (0, \infty)$

Identities

Theorem: $\boxed{\operatorname{sech}^{-1} x = \cosh^{-1} \frac{1}{x}}$

Proof:

$$\operatorname{sech} \left(\cosh^{-1} \left(\frac{1}{x} \right) \right) = \frac{1}{\cosh \left(\cosh^{-1} \left(\frac{1}{x} \right) \right)} = \frac{1}{\frac{1}{x}} = x$$

We consider $x = \operatorname{sech} \left(\cosh^{-1} \left(\frac{1}{x} \right) \right)$ and apply sech^{-1} to both sides.

$$\begin{aligned} x &= \operatorname{sech} \left(\cosh^{-1} \left(\frac{1}{x} \right) \right) \\ \operatorname{sech}^{-1} x &= \operatorname{sech}^{-1} \left(\operatorname{sech} \left(\cosh^{-1} \left(\frac{1}{x} \right) \right) \right) \\ \operatorname{sech}^{-1} x &= \cosh^{-1} \left(\frac{1}{x} \right) \end{aligned}$$

Theorem: $\boxed{\operatorname{csch}^{-1} x = \sinh^{-1} \frac{1}{x}}$

Proof:

$$\operatorname{csch} \left(\sinh^{-1} \left(\frac{1}{x} \right) \right) = \frac{1}{\sinh \left(\sinh^{-1} \left(\frac{1}{x} \right) \right)} = \frac{1}{\frac{1}{x}} = x$$

Now that we have $x = \operatorname{csch} \left(\sinh^{-1} \left(\frac{1}{x} \right) \right)$, we apply csch^{-1} to both sides.

$$\begin{aligned} x &= \operatorname{csch} \left(\sinh^{-1} \left(\frac{1}{x} \right) \right) \\ \operatorname{csch}^{-1} x &= \operatorname{csch}^{-1} \left(\operatorname{csch} \left(\sinh^{-1} \left(\frac{1}{x} \right) \right) \right) \\ \operatorname{csch}^{-1} x &= \sinh^{-1} \left(\frac{1}{x} \right) \end{aligned}$$

Theorem: $\boxed{\operatorname{coth}^{-1} x = \tanh^{-1} \frac{1}{x}}$

Proof:

$$\operatorname{coth} \left(\tanh^{-1} \left(\frac{1}{x} \right) \right) = \frac{1}{\tanh \left(\tanh^{-1} \left(\frac{1}{x} \right) \right)} = \frac{1}{\frac{1}{x}} = x$$

We consider $x = \operatorname{coth} \left(\tanh^{-1} \left(\frac{1}{x} \right) \right)$ and apply coth^{-1} to both sides.

$$\begin{aligned}
 x &= \coth \left(\tanh^{-1} \left(\frac{1}{x} \right) \right) \\
 \coth^{-1} x &= \coth^{-1} \left(\coth \left(\tanh^{-1} \left(\frac{1}{x} \right) \right) \right) \\
 \coth^{-1} x &= \tanh^{-1} \left(\frac{1}{x} \right)
 \end{aligned}$$

Derivatives of inverse functions

Theorem: $\boxed{\frac{d}{dx} \sinh^{-1} x = \frac{1}{\sqrt{1+x^2}}}$

Proof: Recall that when we compose a function with its inverse, the result is the identity function:

$$f(f^{-1}(x)) = x$$

We will state that for $\sinh x$ and differentiate both sides.

$$\begin{aligned}
 \sinh(\sinh^{-1} x) &= x \\
 \cosh(\sinh^{-1} x) \frac{d}{dx}(\sinh^{-1} x) &= 1 \\
 \frac{d}{dx}(\sinh^{-1} x) &= \frac{1}{\cosh(\sinh^{-1} x)}
 \end{aligned}$$

Recall that for all y , $\cosh^2 y - \sinh^2 y = 1$. Then $\cosh y = \pm \sqrt{1 + \sinh^2 y}$. Since $\cosh y$ is always positive, the negative solution is easily ruled out. Thus

$$\frac{d}{dx}(\sinh^{-1} x) = \frac{1}{\cosh(\sinh^{-1} x)} = \frac{1}{\sqrt{1 + \sinh^2(\sinh^{-1} x)}} = \frac{1}{\sqrt{1 + x^2}}$$

Theorem: $\boxed{\frac{d}{dx} \cosh^{-1} x = \frac{1}{\sqrt{x^2 - 1}}}$

Proof: Recall that when we compose a function with its inverse, the result is the identity function:

$$f(f^{-1}(x)) = x$$

We will state that for $\cosh x$ and differentiate both sides. Recall that the domain of $\cosh^{-1} x$ is $[1, \infty)$ (since it is the range of $\cosh x$). On that domain, $\sinh x$ is always positive.

$$\begin{aligned}
 \cosh(\cosh^{-1} x) &= x \\
 \sinh(\cosh^{-1} x) \frac{d}{dx}(\cosh^{-1} x) &= 1 \\
 \frac{d}{dx}(\cosh^{-1} x) &= \frac{1}{\sinh(\cosh^{-1} x)}
 \end{aligned}$$

Recall that for all y , $\cosh^2 y - \sinh^2 y = 1$. Then $\sinh y = \pm \sqrt{\cosh^2 y - 1}$. When $y \geq 1$, $\sinh y$ is positive, and so the negative solution is ruled out. Thus

$$\frac{d}{dx} (\cosh^{-1} x) = \frac{1}{\sinh (\cosh^{-1} x)} = \frac{1}{\sqrt{\cosh^2 (\cosh^{-1} x) - 1}} = \frac{1}{\sqrt{x^2 - 1}}$$

Theorem: $\boxed{\frac{d}{dx} \tanh^{-1} x = \frac{1}{1 - x^2}}$

Proof: Recall that when we compose a function with its inverse, the result is the identity function:

$$f(f^{-1}(x)) = x$$

We will state that for $\tanh x$ and differentiate both sides. Recall that the domain of $\tanh^{-1} x$ is $(-1, 1)$.

$$\begin{aligned} \tanh (\tanh^{-1} x) &= x \\ \operatorname{sech}^2 (\tanh^{-1} x) \frac{d}{dx} (\tanh^{-1} x) &= 1 \\ \frac{d}{dx} (\tanh^{-1} x) &= \frac{1}{\operatorname{sech}^2 (\tanh^{-1} x)} \end{aligned}$$

Recall that for all $\tanh^2 y = 1 - \operatorname{sech}^2 y$. Thus

$$\frac{d}{dx} (\tanh^{-1} x) = \frac{1}{\operatorname{sech}^2 (\tanh^{-1} x)} = \frac{1}{1 - \tanh^2 (\tanh^{-1} x)} = \frac{1}{1 - x^2}$$

Theorem: $\boxed{\frac{d}{dx} \coth^{-1} x = \frac{1}{1 - x^2}}$

Proof: Recall that when we compose a function with its inverse, the result is the identity function:

$$f(f^{-1}(x)) = x$$

We will state that for $\coth x$ and differentiate both sides. Recall that the domain of $\coth^{-1} x$ is $(-\infty, -1) \cup (1, \infty)$.

$$\begin{aligned} \coth (\coth^{-1} x) &= x \\ -\operatorname{csch}^2 (\coth^{-1} x) \frac{d}{dx} (\coth^{-1} x) &= 1 \\ \frac{d}{dx} (\coth^{-1} x) &= -\frac{1}{\operatorname{csch}^2 (\coth^{-1} x)} \end{aligned}$$

Recall that for all $\coth^2 y = 1 + \operatorname{csch}^2 y$. Thus $\operatorname{csch}^2 y = \coth^2 y - 1$ and so

$$\frac{d}{dx} (\coth^{-1} x) = -\frac{1}{\operatorname{csch}^2 (\coth^{-1} x)} = -\frac{1}{\coth^2 (\coth^{-1} x) - 1} = -\frac{1}{x^2 - 1} = \frac{1}{1 - x^2}$$

There is another way to prove this. Recall that $\coth^{-1} x = \tanh^{-1} \left(\frac{1}{x} \right)$

$$\frac{d}{dx} \coth^{-1} x = \frac{d}{dx} \tanh^{-1} \left(\frac{1}{x} \right) = \frac{1}{1 - \left(\frac{1}{x} \right)^2} \left(\frac{d}{dx} \left(\frac{1}{x} \right) \right) = \frac{1}{1 - \left(\frac{1}{x} \right)^2} \left(-\frac{1}{x^2} \right) = \frac{1}{1 - \frac{1}{x^2}} \left(-\frac{1}{x^2} \right) = \frac{-1}{x^2 - 1} = \frac{1}{1 - x^2}$$

Theorem: $\boxed{\frac{d}{dx} \operatorname{sech}^{-1} x = \frac{-1}{x\sqrt{1-x^2}}}$

Proof: Recall that when we compose a function with its inverse, the result is the identity function:

$$f(f^{-1}(x)) = x$$

We will state that for $\operatorname{sech} x$ and differentiate both sides. Recall that the domain of $\operatorname{sech}^{-1} x$ is $[0, 1)$ and range $[0, \infty)$.

$$\begin{aligned} \operatorname{sech}(\operatorname{sech}^{-1} x) &= x \\ -\operatorname{sech}(\operatorname{sech}^{-1} x) \cdot \tanh(\operatorname{sech}^{-1} x) \cdot \frac{d}{dx}(\operatorname{sech}^{-1} x) &= 1 \\ \frac{d}{dx}(\operatorname{sech}^{-1} x) &= -\frac{1}{\operatorname{sech}(\operatorname{sech}^{-1} x) \cdot \tanh(\operatorname{sech}^{-1} x)} = -\frac{1}{x \tanh(\operatorname{sech}^{-1} x)} \end{aligned}$$

Recall that for all $\tanh^2 y = 1 - \operatorname{sech}^2 y$. Thus $\tanh y = \pm \sqrt{1 - \operatorname{sech}^2 y}$. In this case, $\tanh(\operatorname{sech}^{-1} x)$ must be positive and so $\tanh y = \sqrt{1 - \operatorname{sech}^2 y}$. Thus

$$\frac{d}{dx}(\operatorname{sech}^{-1} x) = -\frac{1}{x\sqrt{1 - \operatorname{sech}^2(\operatorname{sech}^{-1} x)}} = -\frac{1}{x\sqrt{1 - x^2}}$$

There is another way to prove this. Recall that $\operatorname{sech}^{-1} x = \cosh^{-1}\left(\frac{1}{x}\right)$

$$\begin{aligned} \frac{d}{dx} \operatorname{sech}^{-1} x &= \frac{d}{dx} \cosh^{-1}\left(\frac{1}{x}\right) = \frac{1}{\sqrt{\left(\frac{1}{x}\right)^2 - 1}} \left(\frac{d}{dx}\left(\frac{1}{x}\right)\right) = \frac{1}{\sqrt{\frac{1}{x^2} - 1}} \left(-\frac{1}{x^2}\right) = \frac{-1}{x^2 \sqrt{\frac{1}{x^2} - 1}} \\ &= \frac{-1}{x \sqrt{x^2 \left(\frac{1}{x^2} - 1\right)}} = \frac{-1}{x \sqrt{1 - x^2}} \end{aligned}$$

Theorem: $\boxed{\frac{d}{dx} \operatorname{csch}^{-1} x = -\frac{1}{|x|\sqrt{1+x^2}}}$

Proof: Recall that when we compose a function with its inverse, the result is the identity function:

$$f(f^{-1}(x)) = x$$

We will state that for $\operatorname{csch} x$ and differentiate both sides. Recall that the domain of $\operatorname{csch}^{-1} x$ is $(-\infty, 0) \cup (0, \infty)$ and range $(-\infty, 0) \cup (0, \infty)$.

$$\begin{aligned} \operatorname{csch}(\operatorname{csch}^{-1} x) &= x \\ -\operatorname{csch}(\operatorname{csch}^{-1} x) \cdot \coth(\operatorname{csch}^{-1} x) \cdot \frac{d}{dx}(\operatorname{csch}^{-1} x) &= 1 \\ \frac{d}{dx}(\operatorname{csch}^{-1} x) &= -\frac{1}{\operatorname{csch}(\operatorname{csch}^{-1} x) \cdot \coth(\operatorname{csch}^{-1} x)} = -\frac{1}{x \coth(\operatorname{csch}^{-1} x)} \end{aligned}$$

Recall that for all $\coth^2 y = 1 + \operatorname{csch}^2 y$. Thus $\coth y = \pm \sqrt{1 + \operatorname{csch}^2 y}$.

$$\frac{d}{dx} (\operatorname{csch}^{-1} x) = -\frac{1}{x \coth (\operatorname{csch}^{-1} x)} = -\frac{1}{\pm x \sqrt{1 + \operatorname{csch}^2 (\operatorname{csch}^{-1} x)}} = \pm \frac{1}{x \sqrt{1 + x^2}}$$

looking at the graph of $\operatorname{csch}^{-1} x$, it is always decreasing and so its derivative must be negative. Thus

$$\frac{d}{dx} (\operatorname{csch}^{-1} x) = -\frac{1}{|x| \sqrt{1 + x^2}}$$

There is another way to prove this. Recall that $\operatorname{csch}^{-1} x = \sinh^{-1} \left(\frac{1}{x} \right)$

$$\begin{aligned} \frac{d}{dx} \operatorname{csch}^{-1} x &= \frac{d}{dx} \sinh^{-1} \left(\frac{1}{x} \right) = \frac{1}{\sqrt{1 + \left(\frac{1}{x} \right)^2}} \left(\frac{d}{dx} \left(\frac{1}{x} \right) \right) = \frac{1}{\sqrt{1 + \frac{1}{x^2}}} \left(-\frac{1}{x^2} \right) = \frac{-1}{x^2 \sqrt{1 + \frac{1}{x^2}}} \\ &= \frac{-1}{|x| \sqrt{x^2 \left(1 + \frac{1}{x^2} \right)}} = \frac{-1}{|x| \sqrt{x^2 + 1}} \end{aligned}$$

Formulas for inverse functions

A more direct formula for $y = \sinh^{-1} x$ exists.

Theorem: $\boxed{\sinh^{-1} x = \ln \left(x + \sqrt{x^2 + 1} \right)}$.

Proof: Consider $y = \sinh^{-1} x$, same as $\sinh y = x$. We will solve this for y .

$$\begin{aligned} x &= \sinh y \\ x &= \frac{1}{2} (e^y - e^{-y}) \\ x &= \frac{1}{2} \left(e^y - \frac{1}{e^y} \right) && \text{multiply by } e^y \\ x e^y &= \frac{1}{2} (e^y)^2 - \frac{1}{2} \end{aligned}$$

This is quadratic in e^y . We will solve for e^y and then for y .

$$\begin{aligned} x e^y &= \frac{1}{2} (e^y)^2 - \frac{1}{2} \\ 0 &= \frac{1}{2} (e^y)^2 - x e^y - \frac{1}{2} \end{aligned}$$

Applying the quadratic formula,

$$e_{1,2}^y = \frac{x \pm \sqrt{x^2 - 4 \left(\frac{1}{2} \right) \left(-\frac{1}{2} \right)}}{2 \left(\frac{1}{2} \right)} = \frac{x \pm \sqrt{x^2 + 1}}{1} = x \pm \sqrt{x^2 + 1}$$

Since $\sqrt{x^2 + 1}$ has a greater absolute value than x , $x - \sqrt{x^2 + 1}$ is always negative and $x + \sqrt{x^2 + 1}$ is always positive. Because e^y is positive for all real y , we discard the negative solution and solve for y .

$$\begin{aligned} e^y &= x + \sqrt{x^2 + 1} \\ y &= \ln \left(x + \sqrt{x^2 + 1} \right) \end{aligned}$$

We can differentiate $\sinh^{-1} x$ using this form of it:

$$\begin{aligned} \frac{d}{dx} \sinh^{-1} x &= \frac{d}{dx} \ln \left(x + \sqrt{x^2 + 1} \right) = \frac{1}{x + \sqrt{x^2 + 1}} \cdot \left(1 + \frac{1}{2\sqrt{x^2 + 1}} (2x) \right) = \frac{1}{x + \sqrt{x^2 + 1}} \cdot \left(1 + \frac{x}{\sqrt{x^2 + 1}} \right) \\ &= \frac{1}{x + \sqrt{x^2 + 1}} \cdot \left(\frac{\sqrt{x^2 + 1} + x}{\sqrt{x^2 + 1}} \right) = \frac{1}{\sqrt{x^2 + 1}} \end{aligned}$$

Similar computation yields for $\boxed{\cosh^{-1} x = \ln \left(x + \sqrt{x^2 - 1} \right)}$.

Now for $\tanh^{-1} x$: Recall that this function has domain $(-1, 1)$.

$$\begin{aligned}
 y &= \tanh^{-1} x \\
 \tanh y &= x \\
 x &= \frac{e^y - e^{-y}}{e^y + e^{-y}} \\
 x(e^y + e^{-y}) &= e^y - e^{-y} \\
 xe^y + x\frac{1}{e^y} &= e^y - \frac{1}{e^y} && \text{multiply by } e^y \\
 x(e^y)^2 + x &= (e^y)^2 - 1 \\
 x + 1 &= (e^y)^2 - x(e^y)^2 \\
 x + 1 &= (e^y)^2(1 - x) \\
 \frac{1+x}{1-x} &= (e^y)^2 \\
 e^y &= \pm \sqrt{\frac{1+x}{1-x}}
 \end{aligned}$$

Since e^y is always positive, we discard the negative solution and solve for y .

$$\begin{aligned}
 e^y &= \sqrt{\frac{1+x}{1-x}} \\
 y &= \ln \sqrt{\frac{1+x}{1-x}} = \frac{1}{2} \ln \left(\frac{1+x}{1-x} \right) = \frac{1}{2} (\ln(1+x) - \ln(1-x))
 \end{aligned}$$

We can differentiate $\tanh^{-1} x$ using this form of it:

$$\begin{aligned}
 \frac{d}{dx} \tanh^{-1} x &= \frac{d}{dx} \left[\frac{1}{2} (\ln(1+x) - \ln(1-x)) \right] = \frac{1}{2} \left(\frac{1}{1+x} - \frac{1}{1-x} (-1) \right) = \frac{1}{2} \left(\frac{1}{1+x} + \frac{1}{1-x} \right) \\
 &= \frac{1}{2} \left(\frac{1-x}{(1+x)(1-x)} + \frac{1+x}{(1-x)(1+x)} \right) = \frac{1}{2} \left(\frac{1-x+1+x}{1-x^2} \right) = \frac{1}{2} \left(\frac{2}{1-x^2} \right) = \frac{1}{1-x^2}
 \end{aligned}$$

Similar formulas exist for all inverse hyperbolic inverse functions:

$$\begin{aligned}
 \sinh^{-1} x &= \ln(x + \sqrt{x^2 + 1}) && x \in \mathbb{R} \\
 \cosh^{-1} x &= \ln(x + \sqrt{x^2 - 1}) && |x| \geq 1 \\
 \tanh^{-1} x &= \frac{1}{2} \ln \left(\frac{1+x}{1-x} \right) = \frac{1}{2} (\ln(1+x) - \ln(1-x)) && |x| < 1 \\
 \operatorname{sech}^{-1} x &= \ln \left(\frac{1 + \sqrt{1-x^2}}{x} \right) && 0 < x \leq 1 \\
 \operatorname{csch}^{-1} x &= \ln \left(\frac{1}{x} + \frac{\sqrt{1+x^2}}{|x|} \right) && x \neq 0 \\
 \operatorname{coth}^{-1} x &= \frac{1}{2} \ln \left(\frac{x+1}{x-1} \right) = \frac{1}{2} (\ln(x+1) - \ln(x-1)) && |x| > 1
 \end{aligned}$$

Formulas

$$\begin{aligned}
\cosh^2 x - \sinh^2 x &= 1 & \sinh 2x &= 2 \sinh x \cosh x & \cosh 2x &= \cosh^2 x + \sinh^2 x \\
\tanh^2 x &= 1 - \operatorname{sech}^2 x & & & &= 1 + 2 \sinh^2 x \\
\coth^2 x &= 1 + \operatorname{csch}^2 x & & & &= 2 \cosh^2 x - 1
\end{aligned}$$

$$\frac{d}{dx} \sinh x = \cosh x \qquad \int \cosh x \, dx = \sinh x + C$$

$$\frac{d}{dx} \cosh x = \sinh x \qquad \int \sinh x \, dx = \cosh x + C$$

$$\frac{d}{dx} \tanh x = \operatorname{sech}^2 x \qquad \int \operatorname{sech}^2 x \, dx = \tanh x + C$$

$$\frac{d}{dx} \coth x = -\operatorname{csch}^2 x \qquad \int \operatorname{csch}^2 x \, dx = -\coth x + C$$

$$\frac{d}{dx} \operatorname{sech} x = -\operatorname{sech} x \tanh x \qquad \int \operatorname{sech} x \tanh x \, dx = -\operatorname{sech} x + C$$

$$\frac{d}{dx} \operatorname{csch} x = -\operatorname{csch} x \coth x \qquad \int \operatorname{csch} x \coth x \, dx = -\operatorname{csch} x + C$$

$$\int \tanh x \, dx = \ln(\cosh x) + C$$

$$\int \coth x \, dx = \ln|\sinh x| + C$$

$$\frac{d}{dx} \sinh^{-1} x = \frac{1}{\sqrt{x^2 + 1}} \qquad \int \frac{1}{\sqrt{x^2 + 1}} \, dx = \sinh^{-1} x + C$$

$$\frac{d}{dx} \cosh^{-1} x = \frac{1}{\sqrt{x^2 - 1}} \qquad x > 1 \qquad \int \frac{1}{\sqrt{x^2 - 1}} \, dx = \cosh^{-1} x + C \qquad x > 1$$

$$\frac{d}{dx} \tanh^{-1} x = \frac{1}{1 - x^2} \qquad |x| < 1 \qquad \int \frac{1}{1 - x^2} \, dx = \tanh^{-1} x + C \qquad |x| < 1$$

$$\frac{d}{dx} \coth^{-1} x = \frac{1}{1 - x^2} \qquad |x| > 1 \qquad \int \frac{1}{1 - x^2} \, dx = \coth^{-1} x + C \qquad |x| > 1$$

$$\frac{d}{dx} \operatorname{sech}^{-1} x = -\frac{1}{x\sqrt{1 - x^2}} \qquad 0 < x < 1 \qquad \int \frac{1}{x\sqrt{1 - x^2}} \, dx = -\operatorname{sech}^{-1} x + C \qquad 0 < x < 1$$

$$\frac{d}{dx} \operatorname{csch}^{-1} x = -\frac{1}{|x|\sqrt{1 + x^2}} \qquad x \neq 0 \qquad \int \frac{1}{|x|\sqrt{1 + x^2}} \, dx = -\operatorname{csch}^{-1} x + C \qquad x \neq 0$$

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