## Sample Problems

We define the **hyperbolic cosine** and **hyperbolic sine** functions as

$$\cosh x = \frac{e^x + e^{-x}}{2} \qquad \text{and} \qquad \sinh x = \frac{e^x - e^{-x}}{2}$$

These functions have interesting properties, very similar to those of the trigonometric functions $\sin x$ and $\cos x$.

1. Prove that $\sinh x$ is an odd function, and $\cosh x$ is an even function.
2. Find the exact values of $\sinh 0$ and $\cosh 0$.
3. Prove that for all $x$, $\cosh^2 x - \sinh^2 x = 1$
4. Can the expression $\cosh^2 x + \sinh^2 x$ be simplified?
5. Prove that $(\sinh x)' = \cosh x$ and $(\cosh x)' = \sinh x$
6. We define $\tanh x = \dfrac{\sinh x}{\cosh x}$.
   (a) Prove that $\tanh x$ is an odd function.
   (b) Prove that $(\tanh x)' = \dfrac{1}{\cosh^2 x}$
7. In trigonometry, we have the formulas $\sin 2x = 2\sin x \cos x$ and $\cos 2x = \cos^2 x - \sin^2 x$. Are there similar true statements about $\sinh 2x$ and $\cosh 2x$?
8. The Saint-Louis arch can be approximated as $y = 715 - 100\cosh\left(\dfrac{x}{100}\right)$. (Units are measured in feet). How tall and how wide is this arch?

 

# Sample Problems - Solutions

We define the **hyperbolic cosine** and **hyperbolic sine** functions as

$$\cosh x = \frac{e^x + e^{-x}}{2} \qquad \text{and} \qquad \sinh x = \frac{e^x - e^{-x}}{2}$$

These functions have interesting properties, very similar to those of the trigonometric functions $\sin x$ and $\cos x$.

1. Prove that $\sinh x$ is an odd function, and $\cosh x$ is an even function.
   Solution:

$$\begin{aligned}\cosh(-x) &= \frac{e^{-x}+e^{-(-x)}}{2} = \frac{e^{-x}+e^{x}}{2} = \cosh x \\ \sinh(-x) &= \frac{e^{-x}-e^{-(-x)}}{2} = \frac{e^{-x}-e^{x}}{2} = -\frac{e^{x}-e^{-x}}{2} = -\sinh x\end{aligned}$$

2. Find the exact values of $\sinh 0$ and $\cosh 0$.
   Solution:

$$\sinh 0 = \frac{e^0 - e^{-0}}{2} = \frac{1-1}{2} = 0 \quad \text{and} \quad \cosh 0 = \frac{e^0 + e^{-0}}{2} = \frac{1+1}{2} = 1$$

3. Prove that for all $x$, $\cosh^2 x - \sinh^2 x = 1$
   Solution:

$$\begin{aligned}\cosh^2 x - \sinh^2 x &= \left(\frac{e^x+e^{-x}}{2}\right)^2 - \left(\frac{e^x-e^{-x}}{2}\right)^2 = \frac{(e^x+e^{-x})^2 - (e^x-e^{-x})^2}{4} \\ &= \frac{e^{2x}+e^{-2x}+2-(e^{2x}+e^{-2x}-2)}{4} = \frac{4}{4} = 1\end{aligned}$$

4. Can the expression $\cosh^2 x + \sinh^2 x$ be simplified?
   Solution, yes, it is $\cosh 2x$. See problem 7.

5. Prove that $(\sinh x)' = \cosh x$ and $(\cosh x)' = \sinh x$
   Solution: The derivative of $e^{-x}$ is $-e^{-x}$ by the chain rule.

$$\begin{aligned}(\sinh x)' &= \left(\frac{e^x-e^{-x}}{2}\right)' = \frac{(e^x-e^{-x})'}{2} = \frac{(e^x-(-e^{-x}))}{2} = \frac{e^x+e^{-x}}{2} = \cosh x \\ (\cosh x)' &= \left(\frac{e^x+e^{-x}}{2}\right)' = \frac{(e^x+e^{-x})'}{2} = \frac{(e^x+(-e^{-x}))}{2} = \frac{e^x-e^{-x}}{2} = \sinh x\end{aligned}$$

 

6. We define $\tanh x = \dfrac{\sinh x}{\cosh x}$

   (a) Prove that $\tanh x$ is an odd function.
   Solution: since $\sinh x$ is odd and $\cosh x$ is even, we easily have

$$\tanh(-x) = \frac{\sinh(-x)}{\cosh(-x)} = \frac{-\sinh x}{\cosh x} = -\tanh x$$

   (b) Prove that $(\tanh x)' = \dfrac{1}{\cosh^2 x}$
   Solution: We will use the quotient rule.

$$\begin{aligned}(\tanh x)' &= \left(\frac{\sinh x}{\cosh x}\right)' = \frac{(\sinh x)' \cosh x - \sinh x (\cosh x)'}{\cos^2 x} = \frac{\cosh x \cosh x - \sinh x \sinh x}{\cos^2 x} \\ &= \frac{\cosh^2 x - \sinh^2 x}{\cos^2 x} = \frac{1}{\cosh^2 x}\end{aligned}$$

7. In trigonometry, we have the formulas $\sin 2x = 2\sin x \cos x$ and $\cos 2x = \cos^2 x - \sin^2 x$. Are there similar true statements about $\sinh 2x$ and $\cosh 2x$?
   Solution:

$$\begin{aligned}\sinh 2x &= \frac{e^{2x} - e^{-2x}}{2} = \frac{(e^x)^2 - (e^{-x})^2}{2} \\ &= \frac{(e^x + e^{-x})(e^x - e^{-x})}{2} = 2 \cdot \frac{(e^x + e^{-x})(e^x - e^{-x})}{4} \\ &= 2 \cdot \frac{(e^x + e^{-x})}{2} \cdot \frac{(e^x - e^{-x})}{2} = 2\sinh x \cosh x\end{aligned}$$

   For a similar statement for $\cosh x$, notice that

$$\begin{aligned}(e^x + e^{-x})^2 &= e^{2x} + e^{-2x} + 2 \quad \text{and} \\ (e^x - e^{-x})^2 &= e^{2x} + e^{-2x} - 2\end{aligned}$$

   and so

$$\begin{aligned}\cosh 2x &= \frac{e^{2x} + e^{-2x}}{2} = \frac{2e^{2x} + 2e^{-2x}}{4} = \frac{(e^x + e^{-x})^2 + (e^x - e^{-x})^2}{4} \\ &= \left(\frac{(e^x + e^{-x})}{2}\right)^2 + \left(\frac{(e^x - e^{-x})}{2}\right)^2 = \cosh^2 x + \sinh^2 x\end{aligned}$$

8. The Saint-Louis arch can be approximated as $y = 715 - 100\cosh\left(\dfrac{x}{100}\right)$. (Units are measured in feet). How tall and how wide is this arch?
   Answer: 530 ft wide, 615 ft tall.

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