

Sample Problems

We define the **hyperbolic cosine** and **hyperbolic sine** functions as

$$\begin{aligned}\cosh x &= \frac{e^x + e^{-x}}{2} & \text{and} & & \sinh x &= \frac{e^x - e^{-x}}{2} \\ \tanh x &= \frac{\sinh x}{\cosh x} & \text{and} & & \coth x &= \frac{\cosh x}{\sinh x}\end{aligned}$$

These functions have interesting properties, very similar to those of the trigonometric functions $\sin x$ and $\cos x$.

1.) Find the exact values of $\sinh 0$ and $\cosh 0$.

Prove each of the following identities.

2.) $\cosh^2 x - \sinh^2 x = 1$

10.) $\coth^2 x - 1 = \operatorname{csch}^2 x$

3.) $1 - \tanh^2 x = \operatorname{sech}^2 x$

11.) $(\sinh x)' = \cosh x$

4.) $\sinh(-x) = -\sinh x$

12.) $(\cosh x)' = \sinh x$

5.) $\cosh(-x) = \cosh x$

13.) $(\tanh x)' = \operatorname{sech}^2 x$

6.) $\sinh 2x = 2 \sinh x \cosh x$

14.) $(\coth x)' = -\operatorname{csch}^2 x$

7.) $\cosh 2x = \cosh^2 x + \sinh^2 x$

15.) $(\operatorname{sech} x)' = -\operatorname{sech} x \tanh x$

8.) $\sinh(x + y) = \sinh x \cosh y + \cosh x \sinh y$

16.) $(\operatorname{csch} x)' = -\operatorname{csch} x \coth x$

9.) $\cosh(x + y) = \cosh x \cosh y + \sinh x \sinh y$

17.) Determine each of the following indefinite integrals.

a) $\int \sinh x dx$ b) $\int \operatorname{sech} x \tanh x dx$ c) $\int \tanh x dx$

18.) The Saint-Louis arch can be approximated as $y = 715 - 100 \cosh\left(\frac{x}{100}\right)$. (Units are measured in feet). How tall and how wide is this arch?

Sample Problems - Solutions

We define the **hyperbolic cosine** and **hyperbolic sine** functions as

$$\begin{aligned}\cosh x &= \frac{e^x + e^{-x}}{2} & \text{and} & & \sinh x &= \frac{e^x - e^{-x}}{2} \\ \tanh x &= \frac{\sinh x}{\cosh x} & \text{and} & & \coth x &= \frac{\cosh x}{\sinh x}\end{aligned}$$

These functions have interesting properties, very similar to those of the trigonometric functions $\sin x$ and $\cos x$.

1. Find the exact values of $\sinh 0$ and $\cosh 0$.

$$\sinh 0 = \frac{e^0 - e^{-0}}{2} = \frac{1 - 1}{2} = 0 \quad \text{and} \quad \cosh 0 = \frac{e^0 + e^{-0}}{2} = \frac{1 + 1}{2} = 1$$

Prove each of the following identities.

2. $\cosh^2 x - \sinh^2 x = 1$

$$\begin{aligned}\cosh^2 x - \sinh^2 x &= \left(\frac{e^x + e^{-x}}{2} \right)^2 - \left(\frac{e^x - e^{-x}}{2} \right)^2 = \frac{(e^x + e^{-x})^2 - (e^x - e^{-x})^2}{4} \\ &= \frac{e^{2x} + e^{-2x} + 2 - (e^{2x} + e^{-2x} - 2)}{4} = \frac{4}{4} = 1\end{aligned}$$

3. $1 - \tanh^2 x = \frac{1}{\cosh^2 x}$

$$1 - \tanh^2 x = 1 - \frac{\sinh^2 x}{\cosh^2 x} = \frac{\cosh^2 x}{\cosh^2 x} - \frac{\sinh^2 x}{\cosh^2 x} = \frac{1}{\cosh^2 x}$$

4. $\sinh(-x) = -\sinh x$

$$\sinh(-x) = \frac{1}{2} (e^{-x} - e^{-(-x)}) = \frac{1}{2} (e^{-x} - e^x) = -\frac{1}{2} (e^x - e^{-x}) = -\sinh x$$

5. $\cosh(-x) = \cosh x$

$$\cosh(-x) = \frac{1}{2} (e^{-x} + e^{-(-x)}) = \frac{1}{2} (e^{-x} + e^x) = \frac{1}{2} (e^x + e^{-x}) = \cosh x$$

6. $\sinh 2x = 2 \sinh x \cosh x$

$$\begin{aligned}\sinh 2x &= \frac{e^{2x} - e^{-2x}}{2} = \frac{(e^x)^2 - (e^{-x})^2}{2} \\ &= \frac{(e^x + e^{-x})(e^x - e^{-x})}{2} = 2 \cdot \frac{(e^x + e^{-x})(e^x - e^{-x})}{4} \\ &= 2 \cdot \frac{(e^x + e^{-x})}{2} \cdot \frac{(e^x - e^{-x})}{2} = 2 \sinh x \cosh x\end{aligned}$$

7. $\cosh 2x = \cosh^2 x + \sinh^2 x$ notice that

$$\begin{aligned}(e^x + e^{-x})^2 &= e^{2x} + e^{-2x} + 2 \quad \text{and} \\ (e^x - e^{-x})^2 &= e^{2x} + e^{-2x} - 2\end{aligned}$$

and so

$$\begin{aligned}\cosh 2x &= \frac{e^{2x} + e^{-2x}}{2} = \frac{2e^{2x} + 2e^{-2x}}{4} = \frac{(e^x + e^{-x})^2 + (e^x - e^{-x})^2}{4} \\ &= \left(\frac{(e^x + e^{-x})}{2}\right)^2 + \left(\frac{(e^x - e^{-x})}{2}\right)^2 = \cosh^2 x + \sinh^2 x\end{aligned}$$

8. $\sinh(x + y) = \sinh x \cosh y + \cosh x \sinh y$

$$\begin{aligned}\sinh x \cosh y + \cosh x \sinh y &= \\ &= \frac{1}{2}(e^x - e^{-x}) \frac{1}{2}(e^y + e^{-y}) + \frac{1}{2}(e^x + e^{-x}) \frac{1}{2}(e^y - e^{-y}) \\ &= \frac{1}{4}(e^x e^y + e^x e^{-y} - e^{-x} e^y - e^{-x} e^{-y}) + \frac{1}{4}(e^x e^y - e^x e^{-y} + e^{-x} e^y - e^{-x} e^{-y}) \\ &= \frac{1}{4}(e^x e^y + e^x e^{-y} - e^{-x} e^y - e^{-x} e^{-y} + e^x e^y - e^x e^{-y} + e^{-x} e^y - e^{-x} e^{-y}) \\ &= \frac{1}{4}(2e^x e^y - 2e^{-x} e^{-y}) = \frac{1}{2}(e^x e^y - e^{-x} e^{-y}) = \frac{1}{2}(e^{x+y} - e^{-(x+y)}) \\ &= \frac{1}{2}(e^{x+y} - e^{-(x+y)}) = \sinh(x + y)\end{aligned}$$

9. $\cosh(x + y) = \cosh x \cosh y + \sinh x \sinh y$

$$\begin{aligned}\cosh x \cosh y + \sinh x \sinh y &= \\ &= \frac{1}{2}(e^x + e^{-x}) \frac{1}{2}(e^y + e^{-y}) + \frac{1}{2}(e^x - e^{-x}) \frac{1}{2}(e^y - e^{-y}) \\ &= \frac{1}{4}(e^x e^y + e^x e^{-y} + e^{-x} e^y + e^{-x} e^{-y}) + \frac{1}{4}(e^x e^y - e^x e^{-y} - e^{-x} e^y + e^{-x} e^{-y}) \\ &= \frac{1}{4}(e^x e^y + e^x e^{-y} + e^{-x} e^y + e^{-x} e^{-y} + e^x e^y - e^x e^{-y} - e^{-x} e^y + e^{-x} e^{-y}) \\ &= \frac{1}{4}(2e^x e^y + 2e^{-x} e^{-y}) = \frac{1}{2}(e^x e^y + e^{-x} e^{-y}) = \frac{1}{2}(e^{x+y} + e^{-(x+y)}) \\ &= \frac{1}{2}(e^{x+y} + e^{-(x+y)}) = \cosh(x + y)\end{aligned}$$

$$10. \coth^2 x - 1 = \frac{1}{\sinh^2 x}$$

$$11. (\sinh x)' = \cosh x$$

$$(\sinh x)' = \left(\frac{e^x - e^{-x}}{2}\right)' = \frac{(e^x - e^{-x})'}{2} = \frac{(e^x - (-e^{-x}))}{2} = \frac{e^x + e^{-x}}{2} = \cosh x$$

12. $(\cosh x)' = \sinh x$

Solution: The derivative of e^{-x} is $-e^{-x}$ by the chain rule.

$$(\cosh x)' = \left(\frac{e^x + e^{-x}}{2} \right)' = \frac{(e^x + e^{-x})'}{2} = \frac{(e^x + (-e^{-x}))}{2} = \frac{e^x - e^{-x}}{2} = \sinh x$$

13. $(\tanh x)' = \frac{1}{\cosh^2 x}$

Solution: We will use the quotient rule.

$$\begin{aligned} (\tanh x)' &= \left(\frac{\sinh x}{\cosh x} \right)' = \frac{(\sinh x)' \cosh x - \sinh x (\cosh x)'}{\cosh^2 x} = \frac{\cosh x \cosh x - \sinh x \sinh x}{\cosh^2 x} \\ &= \frac{\cosh^2 x - \sinh^2 x}{\cosh^2 x} = \frac{1}{\cosh^2 x} \end{aligned}$$

14. $(\coth x)' = -\frac{1}{\sinh^2 x}$

15. $\left(\frac{1}{\cosh x} \right)' = -\frac{1}{\cosh^2 x} \tanh x$

16. $\left(\frac{1}{\sinh x} \right)' = -\frac{1}{\sinh^2 x} \coth x$

17. Determine each of the following indefinite integrals.

a) $\int \sinh x dx = \cosh x + C$

b) $\int \frac{1}{\cosh x} \tanh x dx = -\frac{1}{\cosh^2 x} + C$

c) $\int \tanh x dx = \ln(\cosh x) + C$

18. The Saint-Louis arch can be approximated as $y = 715 - 100 \cosh\left(\frac{x}{100}\right)$. (Units are measured in feet).

How tall and how wide is this arch?

Solution: Define $f(x) = 715 - 100 \cosh\left(\frac{x}{100}\right)$. For the height, we need to find the absolute maximum of f . We find the first derivative of f .

$$f'(x) = -100 \sinh\left(\frac{x}{100}\right) \cdot \frac{1}{100} = -10 \sinh\left(\frac{x}{100}\right)$$

We solve for the zero:

$$\begin{aligned} -10 \sinh\left(\frac{x}{100}\right) &= 0 \\ \sinh\left(\frac{x}{100}\right) &= 0 \\ \frac{e^{x/100} - \frac{1}{e^{x/100}}}{2} &= 0 \\ e^{x/100} &= \frac{1}{e^{x/100}} \\ \left(e^{x/100}\right)^2 &= 1 \\ e^{x/100} &= \pm 1 \end{aligned}$$

$e^{x/100} = -1$ has no solution since $e^{x/100}$ is always positive. To solve for $e^{x/100} = 1$, we take the natural logarithm of both sides.

$$\begin{aligned} e^{x/100} &= 1 \\ \ln(e^{x/100}) &= \ln 1 \\ \frac{x}{100} &= 0 \quad \implies \quad x = 0 \end{aligned}$$

We now compute f'' :

$$\begin{aligned} f'(x) &= -10\,000 \sinh\left(\frac{x}{100}\right) \\ f''(x) &= -10\,000 \cosh\left(\frac{x}{100}\right) \cdot 100 = -1\,000\,000 \cosh\left(\frac{x}{100}\right) \end{aligned}$$

We evaluate f'' at $x = 0$:

$$f''(0) = -1\,000\,000 \cosh\left(\frac{0}{100}\right) = -1\,000\,000$$

Since at $x = 0$ f' is zero and f'' is negative, f has a relative maximum at $x = 0$ by the second derivative test. We evaluate f at $x = 0$:

$$f(0) = 715 - 100 \cosh\left(\frac{0}{100}\right) = 615$$

So the arch is 615 ft tall.

For the width, we need to find x so that $y = 0$.

$$\begin{aligned} f(x) &= 715 - 100 \cosh\left(\frac{x}{100}\right) \\ 0 &= 715 - 100 \cosh\left(\frac{x}{100}\right) \\ \cosh\left(\frac{x}{100}\right) &= 7.15 \\ \frac{e^{x/100} + e^{-x/100}}{2} &= 7.15 \\ e^{x/100} + \frac{1}{e^{x/100}} &= 14.3 \\ e^{x/100} + \frac{1}{e^{x/100}} - 14.3 &= 0 \quad \text{multiply by } e^{x/100} \\ \left(e^{x/100}\right)^2 + 1 - 14.3e^{x/100} &= 0 \\ \left(e^{x/100}\right)^2 - 14.3e^{x/100} + 1 &= 0 \end{aligned}$$

This equation is quadratic in $e^{x/100}$. Let us introduce a new variable, $M = e^{x/100}$. It might be also useful to note that $M > 0$.

$$\begin{aligned} M^2 - 14.3M + 1 &= 0 \\ M_{1,2} &= \frac{14.3 \pm \sqrt{(-14.3)^2 - 4}}{2} = \frac{14.3 \pm \sqrt{200.3}}{2} \approx \begin{cases} 14.229\,724\,571 \\ 0.07027\,542\,91 \end{cases} \end{aligned}$$

Case 1. If $M = 14.229\,724\,571$. Recall that $M = e^{x/100}$

$$\begin{aligned} e^{x/100} &= 14.226\,37 && \text{take natural logarithm} \\ \ln(e^{x/100}) &= \ln(14.229\,724\,571) \\ \frac{x}{100} &= \ln(14.229\,724\,571) && \text{multiply by 100} \\ x &= 100 \ln(14.229\,724\,571) \approx 265.533\,305\,64 \end{aligned}$$

Case 2. If $M = 0.07027\,542\,91$. Recall that $M = e^{x/100}$

$$\begin{aligned} e^{x/100} &= 0.07027\,542\,91 && \text{take natural logarithm} \\ \ln(e^{x/100}) &= \ln(0.07027\,542\,91) \\ \frac{x}{100} &= \ln(0.07027\,542\,91) && \text{multiply by 100} \\ x &= 100 \ln(0.07027\,542\,91) \approx -265.533\,305\,62 \end{aligned}$$

So $(265.533, 0)$ and $(-265.533, 0)$ are the two x -intercepts. Therefore, the arc's width is approximately 531 feet:

$$265.533 - (-265.533) = 531.066$$

For more documents like this, visit our page at <https://teaching.martahidegkuti.com> and click on Lecture Notes. E-mail questions or comments to mhidegkuti@ccc.edu.