Sample Problems

Evaluate each of the following integrals.

1.) $\displaystyle\int_1^\infty \frac{1}{x^4}\,dx$

2.) $\displaystyle\int_1^\infty \frac{1}{x}\,dx$

3.) $\displaystyle\int_{10}^\infty \frac{1}{x\ln x}\,dx$

4.) $\displaystyle\int_0^\infty e^{-5x}\,dx$

5.) $\displaystyle\int_0^\infty xe^{-x^2}\,dx$

6.) $\displaystyle\int_2^\infty \frac{x^2}{\left(x^3-1\right)^4}\,dx$

7.) $\displaystyle\int_2^\infty \frac{1}{\sqrt[3]{2x-1}}\,dx$

Practice Problems

Evaluate each of the following integrals.

1.) $\displaystyle\int_1^\infty \frac{1}{x^2}\,dx$

2.) $\displaystyle\int_1^\infty \frac{1}{\sqrt{x}}\,dx$

3.) $\displaystyle\int_4^\infty \frac{1}{x\left(\ln x\right)^3}\,dx$

4.) $\displaystyle\int_0^\infty x^2e^{-x^3+1}\,dx$

5.) $\displaystyle\int_0^\infty \frac{1}{2}e^{-x/2}\,dx$

6.) $\displaystyle\int_1^\infty \frac{1}{\sqrt{x}\left(\sqrt{x}+1\right)^3}\,dx$

7.) $\displaystyle\int_0^\infty \frac{1}{\sqrt{3x+1}}\,dx$

8.) $\displaystyle\int_0^\infty \frac{1}{\left(3x+1\right)^4}\,dx$

9) $\displaystyle\int_0^\infty \frac{1}{2^{x-1}}\,dx$

Sample Problems - Answers

1.) $\frac{1}{3}$ 2.) ∞ 3.) ∞ 4.) $\frac{1}{5}$ 5.) $\frac{1}{2}$ 6.) $\frac{1}{3087}$ 7.) ∞

Practice Problems - Answers

1.) 1 2.) ∞ 3.) $\frac{1}{2\ln^2 4}$ 4.) $\frac{1}{3}e$ 5.) 1 6.) $\frac{1}{4}$

7.) ∞ 8.) $\frac{1}{9}$ 9) $\frac{2}{\ln 2}$

Sample Problems - Solutions

1. $\displaystyle\int_1^\infty \frac{1}{x^4}dx$

$$\int_1^\infty \frac{1}{x^4}dx = \lim_{N\to\infty}\int_1^N x^{-4}dx = \lim_{N\to\infty}\left.\frac{x^{-3}}{-3}\right|_1^N = \lim_{N\to\infty}\left(\frac{N^{-3}}{-3}-\frac{1^{-3}}{-3}\right) = \lim_{N\to\infty}\left(\frac{-1}{3N^3}-\left(\frac{-1}{3}\right)\right) = \frac{1}{3}$$

2. $\displaystyle\int_1^\infty \frac{1}{x}dx$

$$\int_1^\infty \frac{1}{x}dx = \lim_{N\to\infty}\int_1^N \frac{1}{x}dx = \lim_{N\to\infty}\left.\ln|x|\right|_1^N = \lim_{N\to\infty}(\ln N - \ln 1) = \infty$$

3. $\displaystyle\int_{10}^\infty \frac{1}{x\ln x}dx$

We first compute the indefinite integral, using substitution. Let $u = \ln x$. Then $du = \dfrac{1}{x}dx$ and so $dx = xdu$.

$$\int \frac{1}{x\ln x}dx = \int \frac{1}{xu}xdu = \int \frac{1}{u}du = \ln|u| + C = \ln|\ln x| + C$$

Now we are ready to evaluate the improper integral.

$$\int_{10}^\infty \frac{1}{x\ln x}dx = \lim_{N\to\infty}\int_{10}^N \frac{1}{x\ln x}dx = \lim_{N\to\infty}\left.\ln\ln|x|\right|_{10}^N = \lim_{N\to\infty}(\ln(\ln N) - \ln(\ln 10)) = \infty$$

4. $\displaystyle\int_0^\infty e^{-5x}dx$

$$\int_0^\infty e^{-5x}dx = \lim_{N\to\infty}\int_0^N e^{-5x}dx = \lim_{N\to\infty}\left.\frac{e^{-5x}}{-5}\right|_0^N = \lim_{N\to\infty}\left(\frac{e^{-5N}}{-5}-\frac{e^{-5(0)}}{-5}\right) = \lim_{N\to\infty}\left(\frac{-1}{-5e^{5N}}-\frac{1}{-5}\right) = \frac{1}{5}$$

5. $\displaystyle\int_0^\infty xe^{-x^2}dx$

We first compute the indefinite integral, using substitution. Let $u = -x^2$. Then $du = -2xdx$ and so $dx = \dfrac{du}{-2x}$.

$$\int xe^{-x^2}dx = \int \cancel{x}e^u\frac{du}{-2\cancel{x}} = -\frac{1}{2}\int e^u du = -\frac{1}{2}e^u + C = -\frac{1}{2}e^{-x^2} + C$$

Now we are ready to evaluate the improper integral.

$$\int_0^{\infty} xe^{-x^2}dx = \lim_{N\to\infty}\int_0^{N} xe^{-x^2}dx = \lim_{N\to\infty} -\frac{1}{2}e^{-x^2}\Big|_0^N = -\frac{1}{2}\lim_{N\to\infty}\frac{1}{e^{x^2}}\Big|_0^N = -\frac{1}{2}\lim_{N\to\infty}\left(\frac{1}{e^{N^2}}-\frac{1}{e^{0^2}}\right) = \frac{1}{2}$$

6. $\displaystyle\int_2^{\infty}\frac{x^2}{(x^3-1)^4}\,dx$

We first compute the indefinite integral, using substitution. Let $u = x^3 - 1$. Then $du = 3x^2dx$ and so $dx = \dfrac{du}{3x^2}$.

$$\int \frac{x^2}{(x^3-1)^4}\,dx = \int \frac{x^2}{u^4}\,\frac{du}{3x^2} = \frac{1}{3}\int u^{-4}du = \frac{1}{3}\cdot\frac{u^{-3}}{-3} + C = -\frac{1}{9u^3} + C = \frac{-1}{9(x^3-1)^3} + C$$

Now we are ready to evaluate the improper integral.

$$\begin{aligned}\int_2^{\infty}\frac{x^2}{(x^3-1)^4}\,dx &= \lim_{N\to\infty}\int_2^{N}\frac{x^2}{(x^3-1)^4}\,dx = \lim_{N\to\infty}\frac{-1}{9(x^3-1)^3}\Big|_2^N = -\frac{1}{9}\lim_{N\to\infty}\frac{1}{(x^3-1)^3}\Big|_2^N\\ &= -\frac{1}{9}\lim_{N\to\infty}\left(\frac{1}{(N^3-1)^3}-\frac{1}{(2^3-1)^3}\right) = -\frac{1}{9}\left(-\frac{1}{7^3}\right) = \frac{1}{3087}\end{aligned}$$

7. $\displaystyle\int_2^{\infty}\frac{1}{\sqrt[3]{2x-1}}\,dx$

We first compute the indefinite integral by substitution. Let $u = 2x - 1$. Then $du = 2dx$ and so $dx = \dfrac{du}{2}$.

$$\int\frac{1}{\sqrt[3]{2x-1}}\,dx = \int\frac{1}{\sqrt[3]{u}}\,\frac{du}{2} = \frac{1}{2}\int u^{-1/3}\,du = \frac{1}{2}\frac{u^{2/3}}{\frac{2}{3}} + C = \frac{3}{4}u^{2/3} + C = \frac{3}{4}(2x-1)^{2/3} + C$$

Now we are ready to evaluate the improper integral.

$$\begin{aligned}\int_2^{\infty}\frac{1}{\sqrt[3]{2x-1}}\,dx &= \lim_{N\to\infty}\int_2^{N}\frac{1}{\sqrt[3]{2x-1}}\,dx = \lim_{N\to\infty}\frac{3}{4}(2x-1)^{2/3}\Big|_2^N = \frac{3}{4}\lim_{N\to\infty}(2x-1)^{2/3}\Big|_2^N\\ &= \frac{3}{4}\lim_{N\to\infty}\left((2N-1)^{2/3}-(2\cdot 2-1)^{2/3}\right) = \frac{3}{4}\lim_{N\to\infty}\left((2N-1)^{2/3}-3^{2/3}\right) = \infty\end{aligned}$$

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