Sample Problems

1. (Monomials) Compute each of the following limits.

a) $\lim\limits_{x\to\infty} 3x^4$ c) $\lim\limits_{x\to\infty} \left(-2x^5\right)$ e) $\lim\limits_{x\to\infty} \left(-\frac{2}{3}x^6\right)$ g) $\lim\limits_{x\to\infty} 4x^3$

b) $\lim\limits_{x\to-\infty} 3x^4$ d) $\lim\limits_{x\to-\infty} \left(-2x^5\right)$ f) $\lim\limits_{x\to-\infty} \left(-\frac{2}{3}x^6\right)$ h) $\lim\limits_{x\to-\infty} 4x^3$

2. (Exponential Functions) Compute each of the following limits.

a) $\lim\limits_{x\to\infty} 2^x$ c) $\lim\limits_{x\to\infty} \left(\frac{2}{3}\right)^x$ e) $\lim\limits_{x\to\infty} \frac{2^{x+3}}{3^{x+1}}$ g) $\lim\limits_{x\to\infty} \frac{2^{2x+1}}{3^{x-1}}$

b) $\lim\limits_{x\to-\infty} 2^x$ d) $\lim\limits_{x\to-\infty} \left(\frac{2}{3}\right)^x$ f) $\lim\limits_{x\to-\infty} \frac{2^{x+3}}{3^{x+1}}$ h) $\lim\limits_{x\to-\infty} \frac{2^{2x+1}}{3^{x-1}}$

3. (Basic Functions) Compute each of the following limits.

a) $\lim\limits_{x\to\infty} \frac{1}{x}$

b) $\lim\limits_{x\to-\infty} \frac{1}{x}$

c) $\lim\limits_{x\to\infty} \frac{-5}{2x^3}$

d) $\lim\limits_{x\to-\infty} \frac{3x-2}{x}$

e) $\lim\limits_{x\to\infty} \sqrt{x}$

f) $\lim\limits_{x\to-\infty} \sqrt{x}$

g) $\lim\limits_{x\to\infty} \log_3 x$

h) $\lim\limits_{x\to-\infty} \log_3 x$

i) $\lim\limits_{x\to\infty} \sin x$

j) $\lim\limits_{x\to-\infty} \sin x$

k) $\lim\limits_{x\to\infty} 7$

l) $\lim\limits_{x\to-\infty} 7$

m*) $\lim\limits_{x\to\infty} \left(3^{x+2}-3^x\right)$

n) $\lim\limits_{x\to-\infty} \left(3^{x+2}-3^x\right)$

o*) $\lim\limits_{x\to\infty} \left(\log_{10} 3x - \log_{10} x\right)$

Practice Problems

1. Compute each of the following limits.

a) $\lim\limits_{x\to\infty} \left(-\frac{3}{8}x^{15}\right)$ c) $\lim\limits_{x\to\infty} \frac{1}{3}x^8$ e) $\lim\limits_{x\to\infty} 4x^9$ g) $\lim\limits_{x\to\infty} \left(-7x^{10}\right)$

b) $\lim\limits_{x\to-\infty} \left(-\frac{3}{8}x^{15}\right)$ d) $\lim\limits_{x\to-\infty} \frac{1}{3}x^8$ f) $\lim\limits_{x\to-\infty} 4x^9$ h) $\lim\limits_{x\to-\infty} \left(-7x^{10}\right)$

2. Compute each of the following limits.

a) $\lim\limits_{x\to\infty} \frac{2^{3x-1}}{5^{x-1}}$ c) $\lim\limits_{x\to\infty} \frac{2^{2x+3}}{5^{x-1}}$ e) $\lim\limits_{x\to\infty} \frac{2^{2x+3}}{4^{x-1}}$ g) $\lim\limits_{x\to\infty} \frac{2^{x+3}\cdot 3^{x-1}}{7^{x-2}}$

b) $\lim\limits_{x\to-\infty} \frac{2^{3x-1}}{5^{x-1}}$ d) $\lim\limits_{x\to-\infty} \frac{2^{2x+3}}{5^{x-1}}$ f) $\lim\limits_{x\to-\infty} \frac{2^{2x+3}}{4^{x-1}}$ h) $\lim\limits_{x\to\infty} \frac{2^{2x+3}\cdot 3^{x-1}}{7^{x-2}}$

3. Compute each of the following limits.

a) $\lim_{x\to\infty} \frac{3}{x^5}$

b) $\lim_{x\to-\infty} \frac{3}{x^5}$

c) $\lim_{x\to\infty} \frac{5x-3}{x}$

d) $\lim_{x\to-\infty} \frac{5x-3}{x}$

e) $\lim_{x\to\infty} e^x$

f) $\lim_{x\to-\infty} e^x$

g) $\lim_{x\to\infty} \left(5x - \frac{2}{x+3}\right)$

h) $\lim_{x\to-\infty} \left(5x - \frac{2}{x+3}\right)$

i) $\lim_{\theta\to\infty} \cos 2\theta$

j) $\lim_{\theta\to-\infty} \cos 2\theta$

k) $\lim_{x\to\infty} \frac{5x-3}{x}$

l) $\lim_{x\to-\infty} \frac{5x-3}{x}$

m) $\lim_{x\to\infty} \log_{0.2} x$

n) $\lim_{x\to-\infty} \log_{0.2} x$

o) $\lim_{a\to\infty} \frac{-3a^5+2a-5}{a^2}$

p) $\lim_{a\to-\infty} \frac{-3a^5+2a-5}{a^2}$

q) $\lim_{x\to\infty} (\ln 5x - \ln x)$

r) $\lim_{x\to-\infty} (\ln 5x - \ln x)$

s) $\lim_{x\to\infty} \frac{-4x^8+x^3-x+7}{x^4}$

t) $\lim_{x\to-\infty} \frac{-4x^8+x^3-x+7}{x^4}$

u) $\lim_{x\to\infty} \left(2^{x+1}-2^x\right)$

v) $\lim_{x\to-\infty} \left(2^{x+1}-2^x\right)$

w*) $\lim_{x\to\infty} (\log_2 x - \log_2 (x+1))$

x*) $\lim_{x\to\infty} \left(\log_2\left(\frac{1}{x}\right)\right)$

y*) $\lim_{x\to\infty} \left(\frac{2^x+1}{2^x}\right)$

z) $\lim_{\alpha\to\infty} \tan\alpha$

4. A company is introducing a new product. The marketing manager determines that t weeks after an advertising campaign begins (where $t \geq 4$), $P(t)$ percent of the potential market is aware of the burners, where

$$P(t) = 75\frac{t^2-4t-1}{t^2} + 3 \qquad t \geq 4$$

a) What percent of the potential market knows about the product after 7 weeks?

b) What happens to the percentage $P(t)$ in the long run?

Sample Problems - Answers

1. a) ∞ b) ∞ c) $-\infty$ d) ∞ e) $-\infty$ f) $-\infty$ g) ∞ h) $-\infty$

2. a) ∞ b) 0 c) 0 d) ∞ e) 0 f) ∞ g) ∞ h) 0

3. a) 0 b) 0 c) 0 d) 3 e) ∞ f) undefined g) ∞ f) undefined h) undefined

   i) undefined j) undefined k) 7 l) 7 m) ∞ n) 0 o) $\log_{10} 3$

Practice Problems - Answers

1. a) $-\infty$ b) ∞ c) $\lim_{x\to\infty} \frac{1}{3}x^8 = \infty$ d) ∞ e) ∞ f) $-\infty$ g) $-\infty$ h) $-\infty$

2. a) ∞ b) 0 c) 0 d) ∞ e) 32 f) 32 g) 0 h) ∞

3. a) 0 b) 0 c) 5 d) 5 e) ∞ f) 0 g) ∞ h) $-\infty$ i) undefined j) undefined k) 5 l) 5 m) $-\infty$ n) undefined o) $-\infty$ p) ∞ q) $\ln 5$ r) $\ln 5$ s) $-\infty$ t) $-\infty$

   u) ∞ v) 0 w*) 0 x*) $-\infty$ y*) 1 z) undefined

4. a) $P(7) = \frac{1797}{49} \approx 36.673$ b) $\lim_{t\to\infty} P(t) = 78$ - this means that on the long run, eventually, about 78% of the market will be aware of this product.

Sample Problems - Solutions

1. (Monomials) Compute each of the following limits.

a) $\lim\limits_{x\to\infty} 3x^4$

Solution: Since the limit we are asked for is as x approaches infinity, we should think of x as a very large positive number. Then $3x^4$ is very large, and also positive because it is the product of five positive numbers.

$$3x^4 = \underset{\text{positive}}{3} \cdot \underset{\text{positive}}{x} \cdot \underset{\text{positive}}{x} \cdot \underset{\text{positive}}{x} \cdot \underset{\text{positive}}{x}$$

So the answer is ∞. We state the answer: $\lim\limits_{x\to\infty} 3x^4 = \infty$.

b) $\lim\limits_{x\to-\infty} 3x^4$

Solution: Since the limit we are asked for is as x approaches negative infinity, we should think of x as a very large negative number. Then $3x^4$ is very large, and also positive because it is the product of one positive and four negative numbers.

$$3x^4 = \underset{\text{positive}}{3} \cdot \underset{\text{negative}}{x} \cdot \underset{\text{negative}}{x} \cdot \underset{\text{negative}}{x} \cdot \underset{\text{negative}}{x}$$

So the answer is ∞. We state the answer: $\lim\limits_{x\to-\infty} 3x^4 = \infty$

c) $\lim\limits_{x\to\infty} \left(-2x^5\right)$

Solution: Since the limit we are asked for is as x approaches infinity, we should think of x as a very large positive number. Then $-2x^5$ is very large, and also negative because it is the product of one negative and five positive numbers.

$$-2x^5 = \underset{\text{negative}}{-2} \cdot \underset{\text{positive}}{x} \cdot \underset{\text{positive}}{x} \cdot \underset{\text{positive}}{x} \cdot \underset{\text{positive}}{x} \cdot \underset{\text{positive}}{x}$$

So the answer is $-\infty$. We state the answer: $\lim\limits_{x\to\infty} \left(-2x^5\right) = -\infty$

d) $\lim\limits_{x\to-\infty} \left(-2x^5\right)$

Solution: Since the limit we are asked for is as x approaches negative infinity, we should think of x as a very large negative number. Then $-2x^5$ is very large, and also positive because it is the product of six negative numbers.

$$-2x^5 = \underset{\text{negative}}{-2} \cdot \underset{\text{negative}}{x} \cdot \underset{\text{negative}}{x} \cdot \underset{\text{negative}}{x} \cdot \underset{\text{negative}}{x} \cdot \underset{\text{negative}}{x}$$

So the answer is ∞. We state the answer: $\lim\limits_{x\to-\infty} \left(-2x^5\right) = \infty$

e) $\lim\limits_{x\to\infty} \left(-\dfrac{2}{3}x^6\right)$

Solution: Since the limit we are asked for is as x approaches infinity, we should think of x as a very large positive number. Then $-\dfrac{2}{3}x^6$ is very large, and also negative because it is the product of one negative and six positive numbers.

$$-\frac{2}{3}x^6 = \underset{\text{negative}}{-\frac{2}{3}} \cdot \underset{\text{positive}}{x} \cdot \underset{\text{positive}}{x} \cdot \underset{\text{positive}}{x} \cdot \underset{\text{positive}}{x} \cdot \underset{\text{positive}}{x} \cdot \underset{\text{positive}}{x}$$

So the answer is $-\infty$. We state the answer: $\lim\limits_{x\to\infty}\left(-\frac{2}{3}x^6\right)=-\infty$

f) $\lim\limits_{x\to-\infty}\left(-\frac{2}{3}x^6\right)$

Solution: Since the limit we are asked for is as x approaches negative infinity, we should think of x as a very large negative number. Then $-\frac{2}{3}x^6$ is very large, and also negative because it is the product of seven negative numbers.

$$-\frac{2}{3}x^6 = \underset{\text{negative}}{-\frac{2}{3}} \cdot \underset{\text{negative}}{x} \cdot \underset{\text{negative}}{x} \cdot \underset{\text{negative}}{x} \cdot \underset{\text{negative}}{x} \cdot \underset{\text{negative}}{x} \cdot \underset{\text{negative}}{x}$$

So the answer is $-\infty$. We state the answer: $\lim\limits_{x\to-\infty}-\frac{2}{3}x^6=-\infty$

g) $\lim\limits_{x\to\infty}4x^3$

Solution: Since the limit we are asked for is as x approaches infinity, we should think of x as a very large positive number. Then $4x^3$ is very large, and also positive because it is the product of four positive numbers.

$$4x^3 = \underset{\text{positive}}{4} \cdot \underset{\text{positive}}{x} \cdot \underset{\text{positive}}{x} \cdot \underset{\text{positive}}{x}$$

So the answer is ∞. We state the answer: $\lim\limits_{x\to\infty}4x^3=\infty$

h) $\lim\limits_{x\to-\infty}4x^3$

Solution: Since the limit we are asked for is as x approaches negative infinity, we should think of x as a very large negative number. Then $4x^3$ is very large, and also negative because it is the product of one positive and three negative numbers.

$$4x^3 = \underset{\text{positive}}{4} \cdot \underset{\text{negative}}{x} \cdot \underset{\text{negative}}{x} \cdot \underset{\text{negative}}{x}$$

So the answer is $-\infty$. We state the answer: $\lim\limits_{x\to-\infty}4x^3=-\infty$

2. (Exponential Functions) Compute each of the following limits.

Let $a>0$. Then the limit of the exponential function $f(x)=a^x$ is as follows.

Case 1. If $a>1$, then $\lim\limits_{x\to\infty}a^x=\infty$ and $\lim\limits_{x\to-\infty}a^x=0$

Case 2. If $0<a<1$, then $\lim\limits_{x\to\infty}a^x=0$ and $\lim\limits_{x\to-\infty}a^x=\infty$

a) $\lim\limits_{x\to\infty}2^x$ and b) $\lim\limits_{x\to-\infty}2^x$

Solution: Since $2>1$, these limits are ∞ and 0, i.e. $\lim\limits_{x\to\infty}2^x=\infty$ and $\lim\limits_{x\to-\infty}2^x=0$.

c) $\lim\limits_{x\to\infty}\left(\frac{2}{3}\right)^x$ and d) $\lim\limits_{x\to-\infty}\left(\frac{2}{3}\right)^x$

Solution: Since $\frac{2}{3}<1$, these limits are 0 and ∞, i.e. $\lim\limits_{x\to\infty}\left(\frac{2}{3}\right)^x=0$ and $\lim\limits_{x\to-\infty}\left(\frac{2}{3}\right)^x=\infty$.

e) $\displaystyle\lim_{x\to\infty}\frac{2^{x+3}}{3^{x+1}}$

Solution: We start by re-writing the exponential expressions. The goal is to bring it into a form where there is only one exponential expression involving x.

$$\frac{2^{x+3}}{3^{x+1}}=\frac{2^x\cdot 2^3}{3^x\cdot 3^1}=\frac{2^x\cdot 8}{3^x\cdot 3}=\frac{8}{3}\left(\frac{2}{3}\right)^x$$

$$\text{Thus}\quad \lim_{x\to\infty}\frac{2^{x+3}}{3^{x+1}}=\lim_{x\to\infty}\frac{8}{3}\left(\frac{2}{3}\right)^x=\frac{8}{3}\lim_{x\to\infty}\left(\frac{2}{3}\right)^x=0 \qquad \text{since } \frac{2}{3}<1$$

f) $\displaystyle\lim_{x\to-\infty}\frac{2^{x+3}}{3^{x+1}}$

Solution: $\displaystyle\lim_{x\to-\infty}\frac{2^{x+3}}{3^{x+1}}=\lim_{x\to-\infty}\frac{2^{x+3}}{3^{x+1}}=\lim_{x\to-\infty}\frac{8}{3}\left(\frac{2}{3}\right)^x=\frac{8}{3}\lim_{x\to-\infty}\left(\frac{2}{3}\right)^x=\infty$

g) $\displaystyle\lim_{x\to\infty}\frac{2^{2x+1}}{3^{x-1}}$

Solution: We start by re-writing the exponential expressions. The goal is to bring it into a form where there is only one exponential expression involving x.

$$\frac{2^{2x+1}}{3^{x-1}}=\frac{2^{2x}\cdot 2^1}{\dfrac{3^x}{3^1}}=\frac{\left(2^2\right)^x\cdot 2}{3^x\cdot\dfrac{1}{3}}=\frac{4^x\cdot 6}{3^x}=6\left(\frac{4}{3}\right)^x$$

$$\text{Thus}\quad \lim_{x\to\infty}\frac{2^{2x+1}}{3^{x-1}}=\lim_{x\to\infty}6\left(\frac{4}{3}\right)^x=6\lim_{x\to\infty}\left(\frac{4}{3}\right)^x=\infty \quad \text{since } \frac{4}{3}>1$$

h) $\displaystyle\lim_{x\to-\infty}\frac{2^{2x+1}}{3^{x-1}}$

Solution: $\displaystyle\lim_{x\to-\infty}\frac{2^{2x+1}}{3^{x-1}}=\lim_{x\to-\infty}6\left(\frac{4}{3}\right)^x=6\lim_{x\to-\infty}\left(\frac{4}{3}\right)^x=0$

3. (Basic Functions) Compute each of the following limits.

a) $\displaystyle\lim_{x\to\infty}\frac{1}{x}$

Solution: This is a very important limit. Since the limit we are asked for is as x approaches infinity, we should think of x as a very large positive number. The reciprocal of a very large positive number is a very small positive number. This limit is 0.

b) $\displaystyle\lim_{x\to-\infty} \frac{1}{x}$

Solution: Since the limit we are asked for is as x approaches negative infinity, we should think of x as a very large negative number. The reciprocal of a very large negative number is a very small negative number. This limit is 0.

c) $\displaystyle\lim_{x\to\infty} \frac{-5}{2x^3}$

Solution: Since the limit we are asked for is as x approaches infinity, we should think of x as a very large positive number. We divide -5 by a very large positive number. This limit is 0.

d) $\displaystyle\lim_{x\to-\infty} \frac{3x-2}{x}$

Solution: Now both numerator and denominator approach negative infinity, so we don't know much about the quotient (other than it is positive). But this can be improved by a bit of algebra: we simply divide by x and then the limit becomes much more easy to handle.

$$\lim_{x\to-\infty} \frac{3x-2}{x} = \lim_{x\to-\infty} \left(\frac{3x}{x} - \frac{2}{x}\right) = \lim_{x\to-\infty} \left(3 - \frac{2}{x}\right) = 3$$

As x is a very large negative number, $\dfrac{2}{x}$ approaches zero, and so the limit is 3.

e) $\displaystyle\lim_{x\to\infty} \sqrt{x}$

Solution: The expression $\sqrt{x}$ becomes larger than any fixed number. For example, if x is 1000^2, then x is 1000 and so on. This limit is infinity.

f) $\displaystyle\lim_{x\to-\infty} \sqrt{x}$

Solution: As x becomes a larger and larger negative number, the expression $\sqrt{x}$ is undefined. It is undefined for any negative number. Since there are no function values on the left of zero, there is no limit either. This limit is undefined.

g) $\displaystyle\lim_{x\to\infty} \log_3 x$

The expression $\log_3 x$ becomes larger than any fixed number. For example, if x is 3^{1000}, then x is 1000 and so on. The limit is infinity.

h) $\displaystyle\lim_{x\to-\infty} \log_3 x$

Solution: As x becomes a larger and larger negative number, the expression $\log_3 x$ is undefined. It is undefined for any negative number and also zero. Since there are no function values on the left of zero, there is no limit either. The limit is undefined.

i) $\displaystyle\lim_{x\to\infty} \sin x$

Solution: As x becomes larger and larger, the function values do not zoom in to a fixed value. They just continue to oscillate between -1 and 1 so there is no limit. The answer is undefined.

j) $\lim\limits_{x \to -\infty} \sin x$

Solution: As x becomes a larger and larger negative value, the function values do not zoom in to a fixed value. They just continue to oscillate between -1 and 1 so there is no limit. The answer is undefined.

k) $\lim\limits_{x \to \infty} 7$

Solution: The notation is strange without x ever appearing. This is about the constant function $f(x) = 7$. As x becomes larger and larger, the function values remain 7. So, there is just one value to which they remain very, very, very close: 7. So, the limit is 7.

l) $\lim\limits_{x \to -\infty} 7$

Solution: As x becomes a larger and larger negative value, the function values remain 7. So, there is just one value to which they remain very, very, very close: 7. So, the limit is 7.

m*) $\lim\limits_{x \to \infty} \left(3^{x+2} - 3^x\right)$

Solution: As x becomes larger and larger, both 3^{x+2} and 3^x are very very large, so we do not know much about their difference. The idea that we subtract a very large number from another very large number is not giving us enough clue to determine the limit. This situation called an **indeterminate**. In case of an indeterminate, we cannot evaluate the limit in its original form. We must transform the expression to a form where it is no longer an indeterminate. In this particular example, we need to realize that $3^{x+2} = 3^x \cdot 3^9 = 9 \cdot 3^x$. Then the two exponential expressions can be combined into a single expression that is no longer an indeterminate.

$$\lim_{x \to \infty} \left(3^{x+2} - 3^x\right) = \lim_{x \to \infty} \left(3^x \cdot 3^2 - 3^x\right) = \lim_{x \to \infty} \left(9 \cdot 3^x - 3^x\right) = \lim_{x \to \infty} \left(9 \cdot 3^x - 1 \cdot 3^x\right) = \lim_{x \to \infty} \left(8 \cdot 3^x\right)$$

As x becomes very large, 3^x is extremely large. Then multiplication by 8 just makes it even larger, and so the answer is infinity.

n) $\lim\limits_{x \to -\infty} \left(3^{x+2} - 3^x\right)$

Solution: As x becomes a larger and larger negative number, both 3^{x+2} and 3^x are very very small, so their difference is also very small. Interestingly, this is not an indeterminate. The answer is zero.

Also note that this is not an indeterminate, and we do not need the computation shown above, but it does still work: if we transform our expression to $\lim\limits_{x \to -\infty} \left(8 \cdot 3^x\right)$, we think about an extremely small number (something like 0.0000001) that is multiplied by 8. It will still be extremely small, and so the limit is zero.

o*) $\lim\limits_{x \to \infty} \left(\log_{10} 3x - \log_{10} x\right)$

Solution: As x becomes larger and larger, both $\log_{10} 3x$ and $\log_{10} x$ are very very large, so we do not know much about their difference. The idea that we subtract a very large number from another very large number is not giving us enough clue to determine the limit. This is again an indeterminate. So, we must transform the expression to a form where it is no longer an indeterminate. We will use the following property of logarithms: $\log_{10} a - \log_{10} b = \log_{10} \left(\frac{a}{b}\right)$.

$$\lim_{x \to \infty} \left(\log_{10} 3x - \log_{10} x\right) = \lim_{x \to \infty} \log_{10} \left(\frac{3x}{x}\right) = \lim_{x \to \infty} \log_{10} 3 = \log_{10} 3$$

This expression turned out to be constant, so the limit is the constant value.

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