

Suppose that f and g are differentiable and $g'(x) \neq 0$ on an open interval that contains a (except possibly at a). Suppose that

$$\lim_{x \rightarrow a} f(x) = 0 \text{ and } \lim_{x \rightarrow a} g(x) = 0$$

or that

$$\lim_{x \rightarrow a} f(x) = \pm\infty \text{ and } \lim_{x \rightarrow a} g(x) = \pm\infty$$

(In other words, we have an indeterminate form of type $\frac{0}{0}$ or $\pm\frac{\infty}{\infty}$. Then

$$\lim_{x \rightarrow a} \frac{f(x)}{g(x)} = \lim_{x \rightarrow a} \frac{f'(x)}{g'(x)}$$

If the limit on the right side exists.

Sample Problems

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|---|---|--|
| 1. $\lim_{x \rightarrow \infty} \frac{e^x}{x^2}$ | 6. $\lim_{x \rightarrow 0} \frac{2 \sin x - \sin 2x}{x - \sin x}$ | 11. $\lim_{x \rightarrow 0} \frac{4x - \sin 4x}{x^3}$ |
| 2. $\lim_{x \rightarrow 0} \frac{\cos x - 1}{x + \sin x}$ | 7. $\lim_{x \rightarrow 0^+} x \ln x$ | 12. $\lim_{x \rightarrow 0} \frac{\sin x - \tan x}{x^3}$ |
| 3. $\lim_{x \rightarrow \pi^-} \frac{\sin x}{1 - \cos x}$ | 8. $\lim_{x \rightarrow 0} \frac{5^x - 1}{x^3}$ | 13. $\lim_{x \rightarrow \infty} \frac{\sqrt{x} - \ln x}{\sqrt[3]{x}}$ |
| 4. $\lim_{x \rightarrow 0} \frac{\sin x}{x + 2 \sin x}$ | 9. $\lim_{x \rightarrow 0} \frac{\cos x - 1}{x^2}$ | 14. $\lim_{x \rightarrow 0} \frac{e^{x^2} + 10}{1 - \cos x}$ |
| 5. $\lim_{x \rightarrow \infty} \frac{\sqrt{x}}{\ln x}$ | 10. $\lim_{x \rightarrow 1} \frac{\ln x}{x^2 - x}$ | |

Practice Problems

- | | | |
|--|--|--|
| 1. $\lim_{x \rightarrow 0^+} \frac{e^x - 1}{x^2}$ | 6. $\lim_{m \rightarrow 2} \frac{m^5 - 32}{m^3 - 8}$ | 11. $\lim_{x \rightarrow 0} (x^2 e^{1/x^2})$ |
| 2. $\lim_{x \rightarrow 0} \frac{e^{3x} - 1}{5x}$ | 7. $\lim_{\theta \rightarrow \pi/2} \frac{\tan \theta}{\tan 5\theta}$ | 12. $\lim_{p \rightarrow 0} \frac{e^{3p} - 1}{\sin 2p}$ |
| 3. $\lim_{a \rightarrow 1} \frac{3a^2 - 2a - 1}{5a^2 - a - 4}$ | 8. $\lim_{x \rightarrow \infty} \frac{x}{\ln(x+1)}$ | 13. $\lim_{x \rightarrow 0} \frac{e^{(x^2)} + 10}{1 - \cos x}$ |
| 4. $\lim_{y \rightarrow \infty} \frac{\ln y}{\sqrt[3]{y}}$ | 9. $\lim_{x \rightarrow 0} \frac{x^3}{\tan x - x}$ | 14. $\lim_{x \rightarrow 1} \frac{x^{2/3} - x^{1/2}}{x - 1}$ |
| 5. $\lim_{x \rightarrow \pi} \frac{\sin x}{x - \pi}$ | 10. $\lim_{\beta \rightarrow 0} \frac{\sin \beta - \beta}{\tan \beta - \beta}$ | 15. $\lim_{\alpha \rightarrow 0} \frac{\alpha}{\arctan 2\alpha}$ |

Sample Problems - Answers

- 1.) ∞ 2.) 0 3.) 0 4.) $\frac{1}{3}$ 5.) ∞ 6.) 6 7.) 0 8.) ∞ 9.) $-\frac{1}{2}$ 10.) 1
 11.) $\frac{32}{3}$ 12.) $-\frac{1}{2}$ 13.) ∞ 14.) ∞

Practice Problems - Answers

- 1.) ∞ 2.) $\frac{3}{5}$ 3.) $\frac{4}{9}$ 4.) 0 5.) -1 6.) $\frac{20}{3}$ 7.) 5 8.) ∞ 9.) 3 10.) $-\frac{1}{2}$ 11.) ∞
 12.) $\frac{3}{2}$ 13.) ∞ This is NOT an indeterminate! 14.) $\frac{1}{6}$ 15.) $\frac{1}{2}$

Sample Problems - Solutions

1. $\lim_{x \rightarrow \infty} \frac{e^x}{x^2}$

Solution: This is an $\frac{\infty}{\infty}$ type of an indeterminate, so we can apply l'Hôpital's rule. We differentiate both numerator and denominator:

$$\lim_{x \rightarrow \infty} \frac{e^x}{x^2} = \lim_{x \rightarrow \infty} \frac{e^x}{2x}$$

$\lim_{x \rightarrow \infty} \frac{e^x}{x^2}$ is an $\frac{\infty}{\infty}$ type of an indeterminate, so we can apply l'Hôpital's rule again

$$\lim_{x \rightarrow \infty} \frac{e^x}{2x} = \lim_{x \rightarrow \infty} \frac{e^x}{2} = \boxed{\infty}$$

$\lim_{x \rightarrow \infty} \frac{e^x}{2}$ is no longer an indeterminate because the numerator approaches infinity while the denominator approaches 2. This limit is ∞ .

2. $\lim_{x \rightarrow 0} \frac{\cos x - 1}{x + \sin x}$

Solution: $\lim_{x \rightarrow 0} \frac{\cos x - 1}{x + \sin x}$ is a $\frac{0}{0}$ type of an indeterminate, so we can apply l'Hôpital's rule.

$$\lim_{x \rightarrow 0} \frac{\cos x - 1}{x + \sin x} = \lim_{x \rightarrow 0} \frac{-\sin x}{1 + \cos x} = \frac{0}{2} = \boxed{0}$$

3. $\lim_{x \rightarrow \pi^-} \frac{\sin x}{1 - \cos x}$

Solution: This limit does not qualify for l'Hôpital's rule because substituting π into the expression does NOT result in an indeterminate.

$$\lim_{x \rightarrow \pi^-} \frac{\sin x}{1 - \cos x} = \frac{0}{1 - (-1)} = \frac{0}{2} = \boxed{0}$$

If applied the rule, we would get an incorrect answer, $-\infty$. **So, it is very important to check for the conditions of the rule.**

4. $\lim_{x \rightarrow 0} \frac{\sin x}{x + 2 \sin x}$

Solution: $\lim_{x \rightarrow 0} \frac{\sin x}{x + 2 \sin x}$ a $\frac{0}{0}$ type of an indeterminate, so we can apply l'Hôpital's rule.

$$\lim_{x \rightarrow 0} \frac{\sin x}{x + 2 \sin x} = \lim_{x \rightarrow 0} \frac{\cos x}{1 + 2 \cos x}$$

$\lim_{x \rightarrow 0} \frac{\cos x}{1 + 2 \cos x}$ is no longer an indeterminate: we get $\frac{1}{3}$ when we substitute $x = 0$.

$$\lim_{x \rightarrow 0} \frac{\cos x}{1 + 2 \cos x} = \frac{1}{1 + 2} = \boxed{\frac{1}{3}}$$

Note that we didn't really need L'Hôpital's rule to compute this limit. Recall that $\lim_{x \rightarrow 0} \frac{\sin x}{x} = 1$ and then of course its reciprocal approaches 1 as well: $\lim_{x \rightarrow 0} \frac{x}{\sin x} = 1$. We can compute this limit by dividing both numerator and denominator by $\sin x$.

$$\lim_{x \rightarrow 0} \frac{\sin x}{x + 2 \sin x} = \lim_{x \rightarrow 0} \frac{1}{\frac{x}{\sin x} + 2} = \frac{1}{\lim_{x \rightarrow 0} \frac{x}{\sin x} + 2} = \frac{1}{1 + 2} = \frac{1}{3}$$

5. $\lim_{x \rightarrow \infty} \frac{\sqrt{x}}{\ln x}$

Solution: $\lim_{x \rightarrow \infty} \frac{\sqrt{x}}{\ln x}$ is an $\frac{\infty}{\infty}$ type of an indeterminate, so we can apply l'Hôpital's rule.

$$\lim_{x \rightarrow \infty} \frac{\sqrt{x}}{\ln x} = \lim_{x \rightarrow \infty} \frac{\frac{1}{2\sqrt{x}}}{\frac{1}{x}} = \lim_{x \rightarrow \infty} \frac{1}{2\sqrt{x}} \frac{x}{1} = \lim_{x \rightarrow \infty} \frac{1}{2} \sqrt{x} = \boxed{\infty}$$

6. $\lim_{x \rightarrow 0} \frac{2 \sin x - \sin 2x}{x - \sin x}$

Solution: We substitute x into the expression and get a $\frac{0}{0}$ type of an indeterminate. So we can apply l'Hôpital's rule.

$$\lim_{x \rightarrow 0} \frac{2 \sin x - \sin 2x}{x - \sin x} = \lim_{x \rightarrow 0} \frac{2 \cos x - \cos 2x (2)}{1 - \cos x} = \lim_{x \rightarrow 0} \frac{2 (\cos x - \cos 2x)}{1 - \cos x}$$

This is still a $\frac{0}{0}$ type of an indeterminate, so we can apply l'Hôpital's rule again.

$$\lim_{x \rightarrow 0} \frac{2 (\cos x - \cos 2x)}{1 - \cos x} = \lim_{x \rightarrow 0} \frac{2 (-\sin x + \sin 2x (2))}{\sin x} = \lim_{x \rightarrow 0} \frac{2 (-\sin x + 2 \sin 2x)}{\sin x}$$

This is still a $\frac{0}{0}$ type of an indeterminate, so we can apply l'Hôpital's rule again.

$$\lim_{x \rightarrow 0} \frac{2 (-\sin x + 2 \sin 2x)}{\sin x} = \lim_{x \rightarrow 0} \frac{2 (-\cos x + 2 \cos 2x (2))}{\cos x} = \lim_{x \rightarrow 0} \frac{2 (-\cos x + 4 \cos 2x)}{\cos x}$$

This is no longer an indeterminate:

$$\lim_{x \rightarrow 0} \frac{2 (-\cos x + 4 \cos 2x)}{\cos x} = \frac{2 (-1 + 4)}{1} = \boxed{6}$$

7. $\lim_{x \rightarrow 0^+} x \ln x$

Solution: Although it doesn't appear so, this is either a $\frac{0}{0}$ or an $\frac{\infty}{\infty}$ type of an indeterminate after we re-write it: $\lim_{x \rightarrow 0^+} \frac{\ln x}{\frac{1}{x}}$ is an $\frac{-\infty}{\infty}$ type of an indeterminate and $\lim_{x \rightarrow 0^+} \frac{x}{\frac{1}{\ln x}}$ is a $\frac{0}{0}$ type of an indeterminate. We will use the first form because it seems to yield for slightly easier computation.

$$\lim_{x \rightarrow 0^+} x \ln x = \lim_{x \rightarrow 0^+} \frac{\ln x}{\frac{1}{x}} = \lim_{x \rightarrow 0^+} \frac{\frac{1}{x}}{-\frac{1}{x^2}} = \lim_{x \rightarrow 0^+} \frac{1}{x} \left(-\frac{x^2}{1} \right) = \lim_{x \rightarrow 0^+} (-x) = \boxed{0}$$

8. $\lim_{x \rightarrow 0} \frac{5^x - 1}{x^3}$

Solution: This is a $\frac{0}{0}$ type of an indeterminate, so we can apply l'Hôpital's rule.

$$\lim_{x \rightarrow 0} \frac{5^x - 1}{x^3} = \lim_{x \rightarrow 0} \frac{(\ln 5) 5^x}{3x^2} = \boxed{\infty}$$

In $\lim_{x \rightarrow 0} \frac{(\ln 5) 5^x}{3x^2}$ the numerator approaches 1 while the denominator approaches zero and is positive. This limit is ∞ .

9. $\lim_{x \rightarrow 0} \frac{\cos x - 1}{x^2}$

Solution: This is a $\frac{0}{0}$ type of an indeterminate, so we can apply l'Hôpital's rule.

$$\lim_{x \rightarrow 0} \frac{\cos x - 1}{x^2} = \lim_{x \rightarrow 0} \frac{-\sin x}{2x}$$

This is still a $\frac{0}{0}$ type of an indeterminate, so we can apply l'Hôpital's rule again.

$$\lim_{x \rightarrow 0} \frac{-\sin x}{2x} = \lim_{x \rightarrow 0} \frac{-\cos x}{2} = \boxed{-\frac{1}{2}}$$

10. $\lim_{x \rightarrow 1} \frac{\ln x}{x^2 - x}$

Solution: This is a $\frac{0}{0}$ type of an indeterminate, so we can apply l'Hôpital's rule.

$$\lim_{x \rightarrow 1} \frac{\ln x}{x^2 - x} = \lim_{x \rightarrow 1} \frac{\frac{1}{x}}{2x - 1} = \frac{1}{2 - 1} = \boxed{1}$$

$$11. \lim_{x \rightarrow 0} \frac{4x - \sin 4x}{x^3}$$

Solution: This is a $\frac{0}{0}$ type of an indeterminate, so we can apply l'Hôpital's rule.

$$\lim_{x \rightarrow 0} \frac{4x - \sin 4x}{x^3} = \lim_{x \rightarrow 0} \frac{4 - 4 \cos 4x}{3x^2}$$

This is still a $\frac{0}{0}$ type of an indeterminate, so we can apply l'Hôpital's rule again.

$$\lim_{x \rightarrow 0} \frac{4 - 4 \cos 4x}{3x^2} = \lim_{x \rightarrow 0} \frac{16 \sin 4x}{6x} = \lim_{x \rightarrow 0} \frac{8 \sin 4x}{3x}$$

This is still a $\frac{0}{0}$ type of an indeterminate, so we can apply l'Hôpital's rule again.

$$\lim_{x \rightarrow 0} \frac{8 \sin 4x}{3x} = \lim_{x \rightarrow 0} \frac{32 \cos 4x}{3} = \boxed{\frac{32}{3}}$$

$$12. \lim_{x \rightarrow 0} \frac{\sin x - \tan x}{x^3} = -\frac{1}{2}$$

Solution: This is a $\frac{0}{0}$ type of an indeterminate, so we can apply l'Hôpital's rule.

Recall that $\frac{d}{dx} \tan x = \frac{1}{\cos^2 x}$

$$\lim_{x \rightarrow 0} \frac{\sin x - \tan x}{x^3} = \lim_{x \rightarrow 0} \frac{\cos x - \frac{1}{\cos^2 x}}{3x^2} = \lim_{x \rightarrow 0} \frac{\cos x - (\cos x)^{-2}}{3x^2}$$

This is still a $\frac{0}{0}$ type of an indeterminate, so we can apply l'Hôpital's rule again.

$$\begin{aligned} \lim_{x \rightarrow 0} \frac{\cos x - (\cos x)^{-2}}{3x^2} &= \lim_{x \rightarrow 0} \frac{-\sin x - (-2)(\cos x)^{-3}(-\sin x)}{6x} = \lim_{x \rightarrow 0} \frac{-\sin x - 2 \sin x (\cos x)^{-3}}{6x} \\ &= \lim_{x \rightarrow 0} \frac{\sin x \left(-1 - \frac{2}{\cos^3 x}\right)}{6x} = \lim_{x \rightarrow 0} \frac{\sin x}{x} \lim_{x \rightarrow 0} \frac{\left(-1 - \frac{2}{\cos^3 x}\right)}{6} = 1 \cdot \frac{-1 - 2}{6} = \boxed{-\frac{1}{2}} \end{aligned}$$

If we didn't notice the limit $\lim_{x \rightarrow 0} \frac{\sin x}{x}$ in the expressions, we can still get the answer by applying l'Hôpital's rule one more time.

$$\begin{aligned} \lim_{x \rightarrow 0} \frac{-\sin x - 2 \sin x (\cos x)^{-3}}{6x} &= \\ &= \lim_{x \rightarrow 0} \frac{\sin x \left(-1 - 2(\cos x)^{-3}\right)}{6x} = \lim_{x \rightarrow 0} \frac{\cos x \left(-1 - 2(\cos x)^{-3}\right) + \sin x \left(-2(-3)(\cos x)^{-4}(-\sin x)\right)}{6} \\ &= \lim_{x \rightarrow 0} \frac{\cos x \left(-1 - \frac{2}{\cos^3 x}\right) + \sin x \left(-6 \frac{\sin x}{\cos^4 x}\right)}{6} = \frac{1 \cdot \left(-1 - \frac{2}{1}\right) + 0 \cdot \left(-6 \frac{0}{1}\right)}{6} = -\frac{1}{2} \end{aligned}$$

$$13. \lim_{x \rightarrow \infty} \frac{\sqrt{x} - \ln x}{\sqrt[3]{x}} = \lim_{x \rightarrow \infty} \frac{\frac{1}{2\sqrt{x}} - \frac{1}{x}}{\frac{1}{3}x^{-2/3}} = \lim_{x \rightarrow \infty} \frac{\frac{\sqrt{x}}{2} - 1}{\frac{1}{3}x^{1/3}} = \lim_{x \rightarrow \infty} \frac{\frac{1}{4\sqrt{x}}}{\frac{1}{9}x^{-2/3}} = \lim_{x \rightarrow \infty} \frac{\frac{\sqrt{x}}{4}}{\frac{1}{9}x^{1/3}} = \infty$$

Solution: In case of this problem, it will take some work to determine whether we can use L'Hôpital's rule for this limit. The numerator itself is an $\infty - \infty$ type of an indeterminate. We first factor out $\sqrt{x}$.

$$\lim_{x \rightarrow \infty} (\sqrt{x} - \ln x) = \lim_{x \rightarrow \infty} \sqrt{x} \left(1 - \frac{\ln x}{\sqrt{x}} \right)$$

Inside the parentheses, $\frac{\ln x}{\sqrt{x}}$ is an $\frac{\infty}{\infty}$ type of an indeterminate. We apply l'Hôpital's rule:

$$\lim_{x \rightarrow \infty} \frac{\ln x}{\sqrt{x}} = \lim_{x \rightarrow \infty} \frac{\frac{1}{x}}{\frac{1}{2\sqrt{x}}} = \lim_{x \rightarrow \infty} \frac{1}{x} \frac{2\sqrt{x}}{1} = \lim_{x \rightarrow \infty} \frac{2}{\sqrt{x}} = 0$$

This the numerator is

$$\lim_{x \rightarrow \infty} (\sqrt{x} - \ln x) = \lim_{x \rightarrow \infty} \sqrt{x} \left(1 - \frac{\ln x}{\sqrt{x}} \right) = \lim_{x \rightarrow \infty} \sqrt{x} \lim_{x \rightarrow \infty} \left(1 - \frac{\ln x}{\sqrt{x}} \right) = \left(\lim_{x \rightarrow \infty} \sqrt{x} \right) (1 - 0) = \infty$$

Thus the numerator approaches ∞ , and so we have an $\frac{\infty}{\infty}$ type of an indeterminate. We apply l'Hôpital's rule.

$$\lim_{x \rightarrow \infty} \frac{\sqrt{x} - \ln x}{\sqrt[3]{x}} = \lim_{x \rightarrow \infty} \frac{\frac{1}{2\sqrt{x}} - \frac{1}{x}}{\frac{1}{3}x^{-2/3}}$$

We multiply both numerator and denominator by x :

$$\lim_{x \rightarrow \infty} \frac{\frac{1}{2\sqrt{x}} - \frac{1}{x}}{\frac{1}{3}x^{-2/3}} = \lim_{x \rightarrow \infty} \frac{\frac{\sqrt{x}}{2} - 1}{\frac{1}{3}x^{1/3}}$$

This is still an $\frac{\infty}{\infty}$ type of an indeterminate, so we apply l'Hôpital's rule again.

$$\begin{aligned} \lim_{x \rightarrow \infty} \frac{\frac{\sqrt{x}}{2} - 1}{\frac{1}{3}x^{1/3}} &= \lim_{x \rightarrow \infty} \frac{\frac{1}{4\sqrt{x}}}{\frac{1}{9}x^{-2/3}} = \lim_{x \rightarrow \infty} \frac{\frac{1}{4}x^{-1/2}}{\frac{1}{9}x^{-2/3}} = \lim_{x \rightarrow \infty} \frac{9x^{-1/2-(-2/3)}}{4} = \lim_{x \rightarrow \infty} \frac{9}{4}x^{\frac{2}{3}-\frac{1}{2}} = \lim_{x \rightarrow \infty} \frac{9}{4}x^{\frac{1}{6}} \\ &= \lim_{x \rightarrow \infty} \frac{9}{4} \sqrt[6]{x} = \boxed{\infty} \end{aligned}$$

$$14. \lim_{x \rightarrow 0} \frac{e^{x^2} + 10}{1 - \cos x}$$

This limit can not be solved using l'Hôpital's rule because it is not an indeterminate. The numerator approaches 11, the denominator approaches 0^+ and so the quotient approaches ∞ . If we apply l'Hôpital's rule, we will get a wrong result:

$$\lim_{x \rightarrow 0} \frac{e^{x^2} + 10}{1 - \cos x} = \lim_{x \rightarrow 0} \frac{2xe^{x^2}}{\sin x} = \lim_{x \rightarrow 0} \frac{x}{\sin x} \cdot \lim_{x \rightarrow 0} 2e^{x^2} = 1 \cdot 2 = 2 \quad \text{incorrect}$$

Before applying l'Hôpital's rule, always check first whether the conditions for it hold.