

Sample Problems

1. Compute the trapezoidal approximation for $\int_0^2 \sqrt{x} dx$ using a regular partition with $n = 4$. Compare the estimate with the exact value.
2. Use Simpson's rule to approximate $\int_0^2 \sqrt{x} dx$ using a regular partition with $n = 4$. Compare the estimate with the exact value.

Practice Problems

1. a) Compute the trapezoidal approximation for $\int_0^2 \sqrt{x} dx$ using a regular partition with $n = 6$. Compare the estimate with the exact value.
b) Use Simpson's rule to approximate $\int_0^2 \sqrt{x} dx$ using a regular partition with $n = 6$. Compare the estimate with the exact value.
2. a) Compute the trapezoidal approximation for $\int_0^1 e^{-x^2} dx$ using a regular partition with $n = 10$.
b) Use Simpson's rule to approximate $\int_0^1 e^{-x^2} dx$ using a regular partition with $n = 10$.
3. a) Compute the trapezoidal approximation for $\int_0^1 \frac{1}{1+x^2} dx$ using a regular partition with $n = 8$. Compare the estimate with the exact value.
b) Use Simpson's rule to approximate $\int_0^1 \frac{1}{1+x^2} dx$ using a regular partition with $n = 8$. Compare the estimate with the exact value.

Sample Problems - Answers

1. $T = \frac{1}{4} (2 + 2\sqrt{2} + \sqrt{6}) \approx 1.81948$ The exact value of the integral is $\frac{4\sqrt{2}}{3} \approx 1.88562$

The approximation underestimates the actual area with an error of 0.035076 or 3.51% of the exact value.

2. $\frac{1}{6} (3\sqrt{2} + 2\sqrt{6} + 2) \approx 1.8569367$ The exact value of the integral is $\frac{4\sqrt{2}}{3} \approx 1.88562$

The approximation underestimates the actual area with an error of 0.0152116 or 1.52% of the exact value.

Practice Problems - Answers

1. a) $T \approx 1.84888$ underestimates the area with an error of 0.0194833108 or 1.948% of the area

b) $P \approx 1.86999852$ underestimates the area with an error of 0.008284 or 0.83% of the area.

2. a) $T \approx 0.746210796131749$ b) $P \approx 0.746824948254443$

3. a) $T \approx 0.7847471$ $\int_0^1 \frac{1}{1+x^2} dx = \frac{\pi}{4} \approx 0.7853982$

error: 0.0006511 which is 0.000829 or 0.083% of the exact value

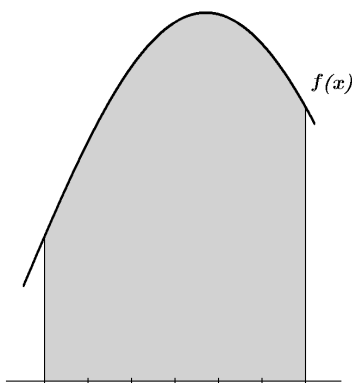
b) $P \approx 0.785398125614677$ $\int_0^1 \frac{1}{1+x^2} dx = \frac{\pi}{4} \approx 0.785398163397448$

error is less than 3.78×10^{-8} which is 4.811×10^{-6} % of the area

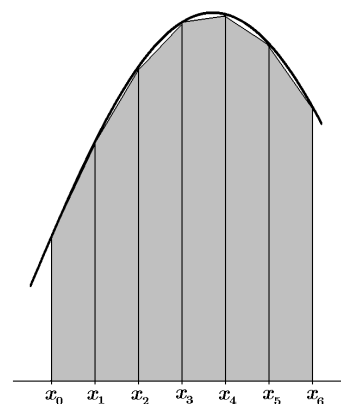
Sample Problems - Solutions

Trapezoidal Approximations

Consider $f(x)$ on the interval $[a, b]$. Define $\Delta x = \frac{b-a}{n}$, in other words, we are looking at a regular partition of $[a, b]$ into n subintervals. In the trapezoidal approximation, the area under the function is approximated by a sum of areas of trapezoids. The picture below shows the trapezoidal approximation of the area under a graph $f(x)$, using a regular partition with $n = 6$.



area under the graph



trapezoidal approximation

Recall that a trapezoid's area is $A = \frac{1}{2} (a + c) h$ where a and c are a pair of parallel sides and h is their distance. Based on our regular partition of $n = 6$, the area of the first trapezoid is $A_1 = \frac{1}{2} (f(x_0) + f(x_1)) \Delta x$, the area of the second trapezoid is $A_2 = \frac{1}{2} (f(x_1) + f(x_2)) \Delta x$, and so on, the area of the last trapezoid is $A_6 = \frac{1}{2} (f(x_5) + f(x_6)) \Delta x$. So the trapezoidal approximation is

$$T = \frac{1}{2} (f(x_0) + f(x_1)) \Delta x + \frac{1}{2} (f(x_1) + f(x_2)) \Delta x + \frac{1}{2} (f(x_2) + f(x_3)) \Delta x + \dots + \frac{1}{2} (f(x_5) + f(x_6)) \Delta x$$

After we factor out $\frac{\Delta x}{2}$, we have

$$T = \frac{\Delta x}{2} (f(x_0) + 2f(x_1) + 2f(x_2) + 2f(x_3) + 2f(x_4) + 2f(x_5) + f(x_6))$$

where the values of x_k can be computed as follows: recall that $\Delta x = \frac{b-a}{6}$ and

$$x_0 = a, \quad x_1 = a + \Delta x, \quad x_2 = a + 2\Delta x, \quad x_3 = a + 3\Delta x, \quad \text{in general, } x_k = a + k\Delta x$$

Example 1: Compute the trapezoidal approximation for $\int_0^2 \sqrt{x} dx$ using a regular partition with $n = 4$. Compare the estimate with the exact value.

Solution: The interval is 2 units long, so with $n = 4$, each subinterval will have length $\frac{1}{2}$. Then

$$x_0 = 0 = \frac{0}{2}, \quad x_1 = \frac{1}{2}, \quad x_2 = \frac{2}{2}, \quad x_3 = \frac{3}{2}, \quad \text{and} \quad x_4 = \frac{4}{2} = 2$$

The trapezoidal approximation is

$$\begin{aligned} T &= \frac{\Delta x}{2} (f(x_0) + 2f(x_1) + 2f(x_2) + 2f(x_3) + f(x_4)) = \frac{1}{4} \left(\sqrt{0} + 2\sqrt{\frac{1}{2}} + 2\sqrt{1} + 2\sqrt{\frac{3}{2}} + \sqrt{2} \right) \\ &= \frac{1}{4} (0 + \sqrt{2} + 2 + \sqrt{6} + \sqrt{2}) = \frac{1}{4} (2 + 2\sqrt{2} + \sqrt{6}) \approx 1.81948 \end{aligned}$$

The exact value of the integral is

$$\int_0^2 \sqrt{x} dx = \int_0^2 x^{1/2} dx = \left. \frac{2}{3} x^{3/2} \right|_0^2 = \frac{2}{3} 2^{3/2} - \frac{2}{3} 0^{3/2} = \frac{2}{3} (2\sqrt{2}) = \frac{4\sqrt{2}}{3} \approx 1.88562$$

The approximation underestimates the actual area, the error is $1.88562 - 1.81948 = 0.06614$ and that is $\frac{0.06614}{1.88562} \approx 0.035076$ or 3.51% of the exact value.

Approximations using parabolas - Simpson's Rule

With this approximation, the regular partition must have an even number of subintervals. In other words, n must be even. We approximate the area under the graph over two consecutive subintervals as the area under the parabola connecting the three endpoints in the two subintervals.

Let (x_{i-1}, y_{i-1}) , (x_i, y_i) and (x_{i+1}, y_{i+1}) will be the three points given to be on the parabola $y = Ax^2 + Bx + C$. We may assume that $x_i = 0$. This is because the area of the parabola does not change if we horizontally shift it until x_i is shifted into the origin. Let h denote the length of the subintervals. In other words, $h = \frac{b-a}{n}$. With this notation, the parabola must pass through the points $(-h, y_{i-1})$, $(0, y_i)$ and (h, y_{i+1}) .

We need to solve for A , B , and C , in terms of x_i , h and y_i . We will have three equations, stating that each point is on the parabola.

$$\begin{aligned} y_{i-1} &= A(-h)^2 + B(-h) + C \\ y_i &= A(0^2) + B(0) + C \\ y_{i+1} &= A(h)^2 + B(h) + C \end{aligned}$$

$$\begin{aligned} y_{i-1} &= Ah^2 - Bh + C \\ y_i &= C \\ y_{i+1} &= Ah^2 + Bh + C \end{aligned}$$

Let us first eliminate the second equation by substituting y_i for C .

$$\begin{aligned} y_{i-1} &= Ah^2 - Bh + y_i \\ y_{i+1} &= Ah^2 + Bh + y_i \end{aligned}$$

$$\begin{aligned} y_{i-1} - y_i &= Ah^2 - Bh \\ y_{i+1} - y_i &= Ah^2 + Bh \end{aligned}$$

Add the two equations to get A :

$$\begin{aligned} y_{i-1} - y_i + y_{i+1} - y_i &= 2Ah^2 \\ \frac{1}{2h^2} (y_{i-1} - 2y_i + y_{i+1}) &= A \end{aligned}$$

Subtract the two equations to get B :

$$\begin{aligned} y_{i+1} - y_i - (y_{i-1} - y_i) &= 2Bh \\ y_{i+1} - y_i - y_{i-1} + y_i &= 2Bh \\ \frac{1}{2h} (y_{i+1} - y_{i-1}) &= B \end{aligned}$$

Thus we have the parabola: $A = \frac{1}{2h^2} (y_{i-1} - 2y_i + y_{i+1})$, $B = \frac{1}{2h} (y_{i+1} - y_{i-1})$, and $C = y_i$.

Now the the area under the parabola is

$$\begin{aligned} P &= \int_{-h}^h (Ax^2 + Bx + C) dx = \left. \frac{A}{3}x^3 + \frac{B}{2}x^2 + Cx \right|_{-h}^h = \left(\frac{A}{3}h^3 + \frac{B}{2}h^2 + Ch \right) - \left(\frac{A}{3}(-h)^3 + \frac{B}{2}(-h)^2 + C(-h) \right) \\ &= \left(\frac{A}{3}h^3 + \frac{B}{2}h^2 + Ch \right) - \left(-\frac{A}{3}h^3 + \frac{B}{2}h^2 - Ch \right) = \frac{A}{3}h^3 + \frac{B}{2}h^2 + Ch + \frac{A}{3}h^3 - \frac{B}{2}h^2 + Ch \\ &= \frac{2}{3}Ah^3 + 2Ch \\ &= \frac{2}{3} \left(\frac{1}{2h^2} (y_{i-1} - 2y_i + y_{i+1}) \right) h^3 + 2(y_i)h \\ &= \frac{1}{3} (y_{i-1} - 2y_i + y_{i+1})h + 2(y_i)h \\ &= \frac{h}{3} (y_{i-1} - 2y_i + y_{i+1} + 6y_i) = \frac{h}{3} (y_{i-1} + 4y_i + y_{i+1}) \end{aligned}$$

The first parabola has area $A_1 = \frac{h}{3} (y_0 + 4y_1 + y_2)$. The second parabola will begin at the same point where the previous one ends, namely in point (x_2, y_2) . Thus $A_2 = \frac{h}{3} (y_2 + 4y_3 + y_4)$. The third parabola has area $A_3 = \frac{h}{3} (y_4 + 4y_5 + y_6)$. And so on, the last parabola has area $A_k = \frac{h}{3} (y_{n-2} + 4y_{n-1} + y_n)$. As we add these areas, the approximation is

$$P = \frac{h}{3} (y_0 + 4y_1 + 2y_2 + 4y_3 + 2y_4 + 4y_5 + \dots + 2y_{n-2} + 4y_{n-1} + y_n)$$

The coefficients are: 1, 4, 2, 4, 2, 4, 2, 4, ..., 2, 4, 1. The same fact can be expressed using the more traditional notation:

$$P = \frac{\Delta x}{3} (f(x_0) + 4f(x_1) + 2f(x_2) + 4f(x_3) + 2f(x_4) + 4f(x_5) + \dots + 2f(x_{n-2}) + 4f(x_{n-1}) + f(x_n))$$

Example 2: Use Simpson's rule to approximate $\int_0^2 \sqrt{x} dx$ using a regular partition with $n = 4$. Compare the estimate with the exact value.

Solution: The interval is 2 units long, so with $n = 4$, each subinterval will have length $\frac{1}{2}$. Then

$$x_0 = 0, \quad x_1 = \frac{1}{2}, \quad x_2 = 1, \quad x_3 = \frac{3}{2}, \quad \text{and} \quad x_4 = 2$$

The approximation with Simpson's rule is

$$\begin{aligned} P &= \frac{\Delta x}{3} (f(x_0) + 4f(x_1) + 2f(x_2) + 4f(x_3) + f(x_4)) = \frac{1}{6} \left(\sqrt{0} + 4\sqrt{\frac{1}{2}} + 2\sqrt{1} + 4\sqrt{\frac{3}{2}} + \sqrt{2} \right) \\ &= \frac{1}{6} (0 + 2\sqrt{2} + 2 + 2\sqrt{6} + \sqrt{2}) = \frac{1}{6} (3\sqrt{2} + 2\sqrt{6} + 2) \approx 1.8569367 \end{aligned}$$

We already computed the exact value of the integral $\int_0^2 \sqrt{x} dx = \frac{4\sqrt{2}}{3} \approx 1.88562$

The approximation underestimates the actual area, the error is $1.88562 - 1.8569367 = 0.0286833$ and that is $\frac{0.0286833}{1.88562} = 0.01521$ or 1.52% of the exact value.

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