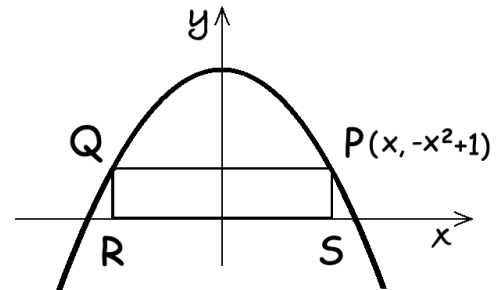


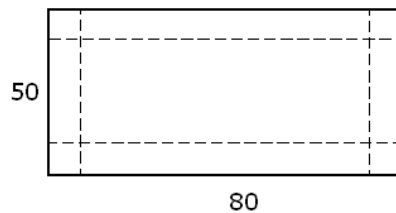
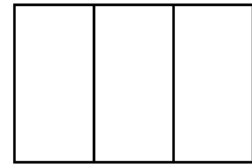
Sample Problems

Part 1 - Polynomials

- Consider $f(x) = -x^4 + 4x^2 + 2$ on the interval $[-3, 3]$.
 - Find all values of x for which f has a relative maximum.
 - Find all values of x for which f has a relative minimum.
 - Find all absolute maximums and minimums of f .
- Let $P(x, y)$ be a point on the graph of $y = -x^2 + 1$ with $0 \leq x \leq 1$. Let $PQRS$ be a rectangle with one side on the x -axis and two vertices on the graph, as shown on the picture next. Find the exact value of the greatest possible area of such a rectangle.

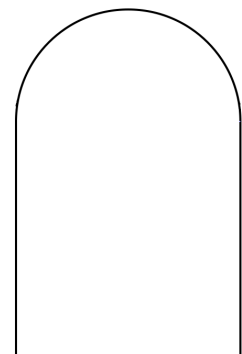


- One positive number plus the square of another equals 48. Choose the numbers so that their product is as large as possible.
- We have P meters of fencing and want to create three adjacent rectangular enclosures as shown on the figure. What is the maximal area we can enclose this way?
- A rectangular box, open at the top, is to be constructed from a rectangular sheet of cardboard 50 centimeters by 80 centimeters by cutting out equal squares in the corners and folding up the sides. What sides squares should be cut out for the container to have maximal volume?



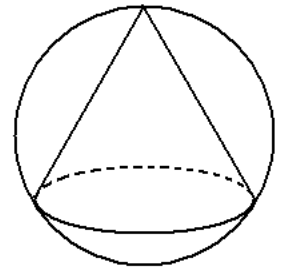
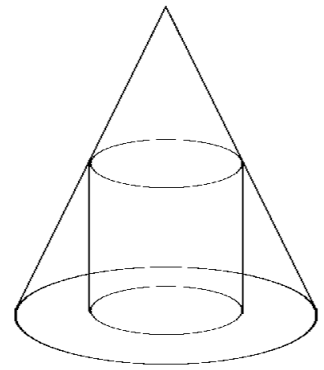
- Prove that for any real numbers a and b , if $a + b = 1$, then $a^4 + b^4 \geq \frac{1}{8}$.

- One thousand feet of fencing is to be used to surround two areas, one square and one circular. What should be the size of each area so that the total area is
 - as large as possible
 - as small as possible?
- A Norman window has the outline of a semicircle on top of a rectangle, as shown on the picture. Find the dimensions of the window that can be built using 8 meters of wood and has the maximal area.



- Find the point(s) on the arc of the parabola $y = x^2$ nearest to the point $(0, 2)$.

10. A company wants to manufacture cylindrical aluminum cans using 100cm^2 of aluminum. What dimensions would guarantee the maximal volume?
11. We write a cylinder into a cone as shown on the picture. The cone is of height 15cm and its base has a radius of 6cm . What dimensions of the cylinder would guarantee the greatest volume?
12. Consider a sphere with radius R and all right cones we can write into it as shown on the picture. Which one has the greatest volume?



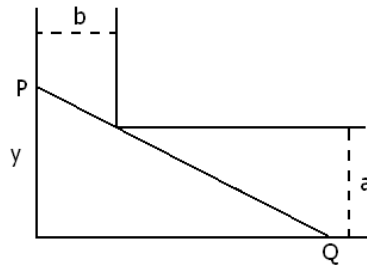
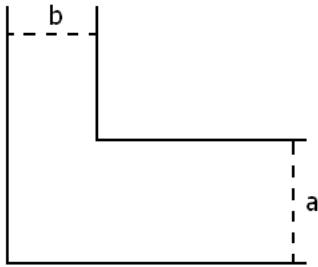
Part 2 - Generalized Polynomials

1. We are designing a poster to contain 50in^2 of printing with a 4-inch wide margin at the top and bottom and a 2-inch wide margin at each side. What overall dimensions will minimize the amount of paper used?
2. Let a and b be positive numbers such that $ab = 10$. Find the lowest value of $a^2 + 4b^2$.
1. A closed box with a square base is to have a volume of 250 cubic meters. The material for the top and bottom of the box costs \$2 per square meter, and the material for the sides costs \$1 per square meter. Can the box be constructed for less than \$300?
2. A company wants to manufacture cylindrical aluminum cans with a volume of 1000 cubic centimeters (one liter). What dimensions would guarantee the minimal amount of aluminum needed to produce a can?

Part 3 - After the Chain Rule

1. A company determines that if n is the number of items produced, they can all be sold at a price of $p(n) = \sqrt{1200 - 0.2n}$. What is the greatest revenue possible?
2. A company has \$120 000 to spend on the development and promotion of a new product. The company estimates that if x is spent on the development and y is spent on promotion, then approximately $\frac{x^{1/2}y^{3/2}}{400\,000}$ items of new product will be sold. Based on this estimate, what is the maximum number of products that the company can sell?
3. An underground telephone cable is to be laid between two boat docks on opposite banks of a straight river. One boathouse is 600 meters downstream from the other. The river is 200 meters wide. If the cost of laying the cable is \$50 per meter under water and \$30 per meter on land, how should the cable be laid to minimize cost?
 - a) Find the lowest cost possible.
 - b) Use the second derivative test to prove that we found a relative minimum in part a).
 - c) Prove that the relative minimum we found in part a) is also an absolute minimum on the domain $[0, 600]$.
 - d) Prove that if we define C on the set of all real numbers, then the minimum we found in part a) is still an absolute minimum.

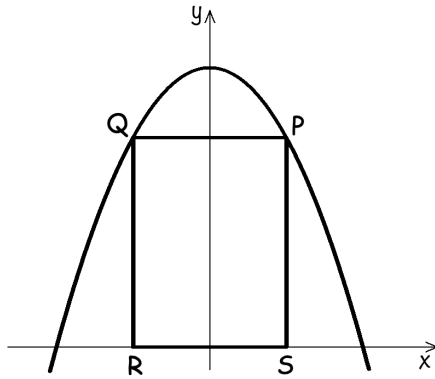
4. The location function of an object is $s(t) = \frac{120}{1 + 2e^{-t}}$ where t is time, measured in seconds. Where is the object when it is moving with the greatest speed? When is that greatest speed achieved?
5. Consider the function $g(x) = \frac{-2x}{(x^2 + 1)^2}$.
- Find all relative extrema of g .
 - Find all absolute extrema of g .
 - Find all values of c for which the function $f(x) = \frac{1}{x^2 + 1} + cx$ is increasing on its entire domain.
6. Consider the problem of finding the length of the longest rod we can carry through a corner shown on the picture.
- For what values of y does the line segment PQ have the shortest length? (Hint: minimize the square of the length.)
 - What is the length of the longest ladder which can be slid along the floor from a corridor of width a to a corridor of width b ?



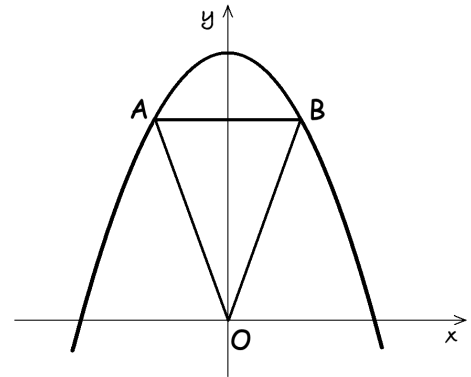
Practice Problems

- Consider $f(x) = x^3 + 3x^2 - 9x - 12$ on the interval $[-4, 4]$.
 - Find all values of x for which f has a relative maximum.
 - Find all values of x for which f has a relative minimum.
 - Find all absolute maximums and minimums of f .
- Consider $f(x) = x^3 - 3x^2 + 6$ on the interval $[-1, 4]$.
 - Find all values of x for which f has a relative maximum.
 - Find all values of x for which f has a relative minimum.
 - Find all absolute maximums and minimums of f .

3. a) Let $P(x, y)$ be a point on the graph of $y = 4 - x^2$ with $0 \leq x \leq 2$. Let $PQRS$ be a rectangle with one side on the x -axis and two vertices on the graph, as shown on the picture below. Find the exact value of the coordinates of P and of the greatest possible area of such a rectangle.

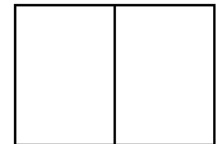


- b) Let $B(x, y)$ be a point on the graph of $y = 1 - x^2$ with $0 \leq x \leq 1$. Let ABO be an isosceles triangle with one horizontal side and third vertex at the origin, as shown on the picture below. Find the exact value of the coordinates of B and of the greatest possible area of such a triangle.

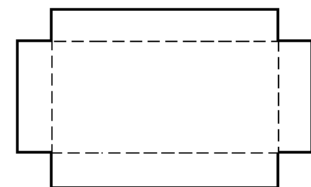


Part 1 - Polynomials

4. a) One positive number plus the square of another equals 15. Choose the numbers so that their product is as large as possible.
- b) Suppose that $S > 0$. One positive number plus the square of another equals S . Choose the numbers so that their product is as large as possible.
5. We have P meters of fencing and want to create two adjacent rectangular enclosings as shown on the figure below. What is the maximal area we can enclose this way? What dimensions will guarantee this maximal area?

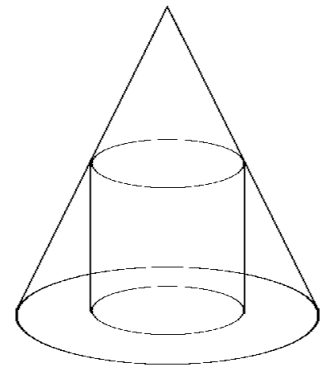


6. a) A rectangular box, open at the top, is to be constructed from a rectangular sheet of cardboard 12 centimeters by 24 centimeters by cutting out equal squares in the corners and folding up the sides. What sides squares should be cut out for the container to have maximal volume?

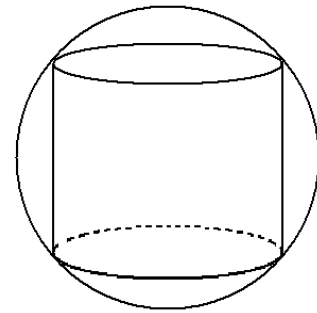


- b) Suppose that $A, B > 0$. A rectangular box, open at the top, is to be constructed from a rectangular sheet of cardboard A centimeters by B centimeters by cutting out equal squares in the corners and folding up the sides. What sides squares should be cut out for the container to have maximal volume?
7. a) Find the point(s) on the line $3x + y = 12$ nearest to the origin.
- b) Find the point(s) on the line $3x + y = 12$ nearest to the point $(2, 10)$.
8. a) Find the point(s) on the graph of $y = x^2$ nearest to the point $(0, 3)$.
- b) Suppose that $B > 0$. Find the point(s) on the graph of $y = x^2$ nearest to the point $(0, B)$.

9. a) We write a cylinder into a cone as shown on the picture below. The cone is of height 24cm and its base has a radius of 12cm. What dimensions of the cylinder would guarantee the greatest volume?
- b) We write a cylinder into a cone as shown on the picture below. The cone is of height H and its base has a radius of R . What dimensions of the cylinder would guarantee the greatest volume?



1. Consider all right cylinders written into a sphere with radius R as shown on the picture. What dimensions (height and radius) will guarantee the greatest volume for the cylinder?



Part 2 - Generalized Polynomials

1. We are designing a poster to contain 60in^2 of printing with a 3-inch wide margin at the top and bottom and a 2-inch wide margin at each side. What overall dimensions will minimize the amount of paper used?
1. Let x and y be positive numbers such that $xy = 1$. Find the lowest possible value of $x^3 + 2y^3$.
2. We would like to construct an open box with a square base. We would like the box to have a volume of 200in^3 . What dimensions would guarantee that the box can be made using the least amount of material?
3. A company wants to manufacture cylindrical aluminum cans with a volume of 1000 cubic centimeters (one liter). The material for the base costs 20 cents per cm^2 and the material for the sides costs 10 cents per cm^2 . What dimensions would guarantee the minimal cost of producing such a can?



Answers

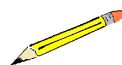
Sample Problems

1. a) $-\sqrt{2}, \sqrt{2}$ b) 0 c) absolute minimum: $(-3, -43)$ and $(3, -43)$; absolute maximum: $(-\sqrt{2}, 6)$ and $(\sqrt{2}, 6)$
2. $\frac{4\sqrt{3}}{9}$ 3. 4 and 32 4. $\frac{P^2}{32}$ 5. 10 cm by 10 cm 6. see solutions
7. a) $r_{\max} = \frac{500}{\pi}$ and $s_{\max} = 0$ b) $r_{\min} = \frac{500}{4 + \pi}$ and $s_{\min} = \frac{1000}{\pi + 4}$ 8. width: $\frac{16}{\pi + 4}$ height: $\frac{16}{\pi + 4}$
9. $\left(-\sqrt{\frac{3}{2}}, \frac{3}{2}\right)$ and $\left(\sqrt{\frac{3}{2}}, \frac{3}{2}\right)$ 10. $r = \sqrt{\frac{50}{3\pi}}$ and $h = 2\sqrt{\frac{50}{3\pi}}$
11. $V_{\max} = 80\pi\text{cm}^3$, when radius is 4cm and height 5cm 12. $h = \frac{4}{3}R$ $r = \frac{2\sqrt{2}}{3}R$ and $V_{\max} = \frac{32}{81}\pi R^3$
13. 9 inches by 18 inches 14. 40
15. no, the lowest possible cost is \$300 when the box is to be 5m by 5m by 10m
16. $r = \sqrt[3]{\frac{500}{\pi}} \approx 5.41926$ and $h = 2\sqrt[3]{\frac{500}{\pi}} = 2r \approx 10.83852$ 17. \$80 000 18. 11 691
19. a) \$26 000 when the cable is laid 450 meters on the ground b) see solutions c) see solutions d) see solutions
20. $s = 60$ when $t = \ln 2$ 21. a) $\left(-\frac{1}{\sqrt{3}}, \frac{3\sqrt{3}}{8}\right)$ relative maximum and a relative minimum $\left(-\frac{1}{\sqrt{3}}, \frac{3\sqrt{3}}{8}\right)$ b) see solutions
- c) $c \geq \frac{3\sqrt{3}}{8}$ 22. a) $y = \sqrt[3]{ab^2} + a$ b) $(a^{2/3} + b^{2/3})^{3/2} = \sqrt{\left(\sqrt[3]{a^2} + \sqrt[3]{b^2}\right)^3}$

Practice Problems

1. a) -3 b) 1 c) absolute maximum: (4, 64) absolute minimum: (1, -17)
2. a) 0 b) 2 c) absolute maximum: (4, 22) absolute minimum: (-1, 2) and (2, 2)
3. a) $P\left(\frac{2\sqrt{3}}{3}, \frac{8}{3}\right)$ Area = $\frac{32\sqrt{3}}{9}$ b) $B\left(\frac{\sqrt{3}}{3}, \frac{2}{3}\right)$ Area = $\frac{2\sqrt{3}}{9}$
4. a) $\sqrt{5}, 10$ b) $\sqrt{\frac{S}{3}}, \frac{2}{3}S$ 5. horizontal side: $\frac{P}{4}$ vertical side: $\frac{P}{6}$ Area: $\frac{P^2}{24}$
6. a) $6 + 2\sqrt{3}$ b) $\frac{A + B + \sqrt{A^2 - AB + B^2}}{6}$ 7. a) $\left(\frac{18}{5}, \frac{6}{5}\right)$ b) (4, 0)
8. a) $\left(-\sqrt{\frac{5}{2}}, \frac{5}{2}\right)$ and $\left(\sqrt{\frac{5}{2}}, \frac{5}{2}\right)$
b) If $B \leq \frac{1}{2}$, then the origin is closest. If $B > \frac{1}{2}$, then $\left(-\sqrt{B - \frac{1}{2}}, B - \frac{1}{2}\right)$ and $\left(\sqrt{B - \frac{1}{2}}, B - \frac{1}{2}\right)$
9. a) 8cm radius, 8cm height, $V_{\max} = 512\pi\text{cm}^3$ b) $\frac{2}{3}R$ radius, $\frac{1}{3}H$ height, $V_{\max} = \frac{4}{27}\pi R^2 H$
10. $h = \frac{2\sqrt{3}}{3}R$ and $r = \frac{\sqrt{6}}{3}R$ $V_{\max} = \frac{4\sqrt{3}}{9}\pi R^3$ 11. $2\sqrt{10}$ inches by $3\sqrt{10}$ inches 12. $2\sqrt{2}$ (when $x = \sqrt[6]{2}$)
13. base: $\sqrt[3]{400}\text{in}$ by $\sqrt[3]{400}\text{in}$ height: $\frac{1}{2}\sqrt[3]{400}\text{in}$ 14. $r = 5\sqrt[3]{\frac{2}{\pi}}\text{cm} \approx 4.30127\text{cm}$ and $h = \frac{20}{\pi}\sqrt[3]{\frac{2}{\pi}} = 5.476547\text{cm}$

Sample Problems



Solutions

1. Consider $f(x) = -x^4 + 4x^2 + 2$ on the interval $[-3, 3]$.

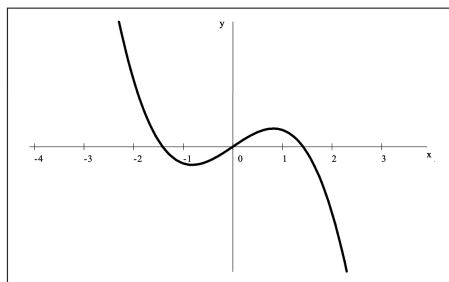
a) Find all values of x for which f has a relative maximum.

Solution: We differentiate f and sketch the derivative f' . All relative maximums will be where the derivative f' changes sign from positive to negative, provided that f is continuous there. Since f is a polynomial, it is continuous on its entire domain and so we will find the maximum where f' changes sign from positive to negative.

$$\begin{aligned} f(x) &= -x^4 + 4x^2 + 2 \\ f'(x) &= -4x^3 + 8x = -4x(x^2 - 2) = -4x(x + \sqrt{2})(x - \sqrt{2}) \end{aligned}$$

We sketch the graph of the derivative f' .

Since the derivative f' changes sign from positive to negative at $x = -\sqrt{2}$ and $x = \sqrt{2}$, f has relative maximums there.

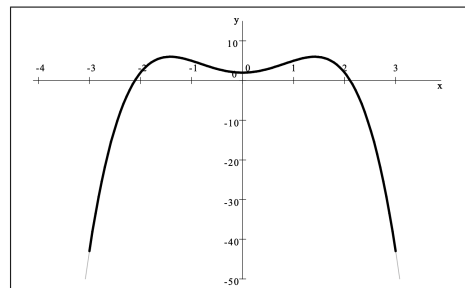


b) Find all values of x for which f has a relative minimum.

Solution: All relative minimums will be where the derivative f' changes sign from negative to positive, provided that f is continuous there. Since f is a polynomial, it is continuous on its entire domain and so we will find the relative minimum where f' changes sign from negative to positive. We sketched the graph of f' in part a) and so it is easy to see that f has a relative minimum at $x = 0$.

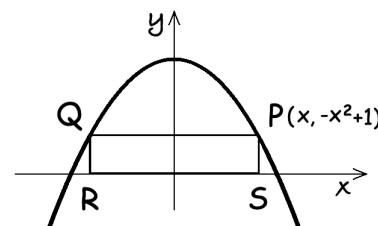
c) Find all absolute maximums and minimums of f .

Solution: If a function is continuous on a closed interval, it always achieves an absolute maximum and minimum. The candidates for these are the relative extrema and the endpoints of the domain. Thus, all we have to do is to evaluate f at $x = -3, -\sqrt{2}, 0, \sqrt{2}, 3$. (Note that if we notice that f is an even function, that can save us a few computations.) We compute that $f(-3) = -43$, $f(-\sqrt{2}) = 6$, $f(0) = 2$, $f(\sqrt{2}) = 6$, and $f(3) = -43$. Thus f has absolute minimums $(-3, -43)$ and $(3, -43)$ and absolute maximums $(-\sqrt{2}, 6)$ and $(\sqrt{2}, 6)$.



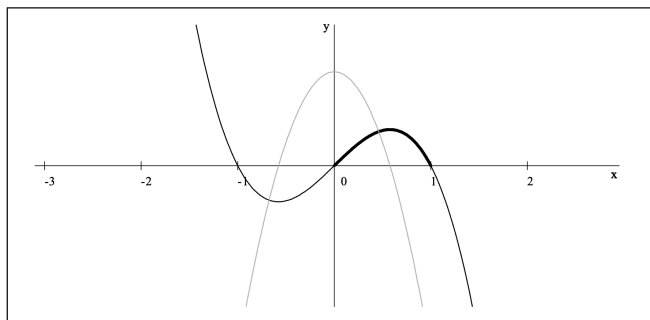
2. Let $P(x, y)$ be a point on the graph of $y = -x^2 + 1$ with $0 \leq x \leq 1$. Let $PQRS$ be a rectangle with one side on the x -axis and two vertices on the graph, as shown on the picture below. Find the exact value of the greatest possible area of such a rectangle.

Solution: Let P be the point with coordinates $(x, -x^2 + 1)$. For the rectangle to have non-zero sides, $0 < x < 1$. Using the notation described in the problem, the horizontal side of the rectangle is $2x$ long, and the vertical side is $-x^2 + 1$ long. Thus the area of the rectangle, as a function of x , is $A(x) = 2x(-x^2 + 1) = -2x^3 + 2x$, on domain $(0, 1)$. The derivative is $A'(x) = -6x^2 + 2$.



$$A'(x) = -6x^2 + 2 = -6\left(x^2 - \frac{1}{3}\right) = -6\left(x + \frac{1}{\sqrt{3}}\right)\left(x - \frac{1}{\sqrt{3}}\right)$$

which is a downward opening parabola, with x -intercepts at $x = -\frac{1}{\sqrt{3}}$ and $x = \frac{1}{\sqrt{3}}$. Based on the sign of A' , we determine that A has a relative minimum at $x = -\frac{1}{\sqrt{3}}$ and a relative maximum at $x = \frac{1}{\sqrt{3}}$. However, the domain of our function is $(0, 1)$. If we sketch A on the domain $(0, 1)$, it is clear from the graph that $A(x)$ has an absolute maximum at $x = \frac{1}{\sqrt{3}}$



Thus the greatest area is $A\left(\frac{1}{\sqrt{3}}\right)$. We compute this number:

$$A\left(\frac{1}{\sqrt{3}}\right) = 2\left(\frac{1}{\sqrt{3}}\right)\left(-\left(\frac{1}{\sqrt{3}}\right)^2 + 1\right) = 2\left(\frac{\sqrt{3}}{3}\right)\left(-\frac{1}{3} + 1\right) = 2\left(\frac{\sqrt{3}}{3}\right)\left(\frac{2}{3}\right) = \frac{4\sqrt{3}}{9}$$

And so the greatest possible area is $\frac{4\sqrt{3}}{9}$.

3. One positive number plus the square of another equals 48. Choose the numbers so that their product is as large as possible.

Solution: Let us denote the numbers by a and b . We have $a + b^2 = 48 \implies a = 48 - b^2$. Then the product of the two numbers is a function of b ,

$$P(b) = b(48 - b^2) = -b^3 + 48b$$

Since this is a cubic function, it does not have an absolute maximum, only if its domain is restricted.

The domain is

b is positive $\implies b > 0$ and

$a = 48 - b^2$ is positive.

We solve $48 - b^2 > 0$ for b :

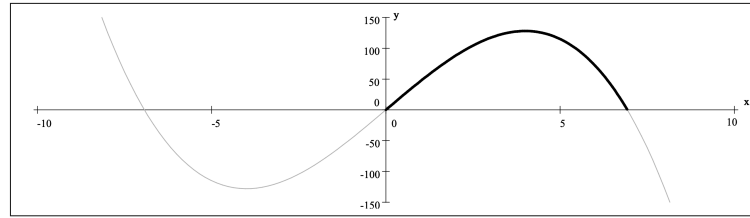
$$48 - b^2 > 0$$

$$48 > b^2$$

$$-\sqrt{48} < b < \sqrt{48}$$

So, a is positive when b is between $-\sqrt{48}$ and $\sqrt{48}$, which means that both a and b are positive when b is between 0 and $\sqrt{48}$. Therefore, the domain of P is $(0, \sqrt{48})$.

We can sketch P ; it is a cubic polynomial with a negative leading coefficient, and its zeroes are at $x = 0, -\sqrt{48}, \sqrt{48}$ as its factored form is $P(b) = -b(b^2 - 48) = -b(b + \sqrt{48})(b - \sqrt{48})$

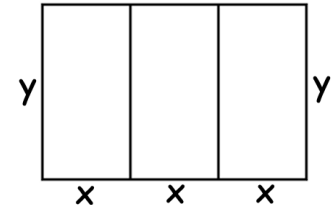


The picture shows that P will have a **relative** maximum on the interval $(0, \sqrt{48})$ that is **also absolute maximum**. We find its x -coordinate by solving for the zero of the derivative.

$$\begin{aligned} P(b) &= -b^3 + 48b \\ P'(b) &= -3b^2 + 48 = -3(b^2 - 16) = -3(b + 4)(b - 4) \end{aligned}$$

Thus the maximum is at $b = 4$. Then the other number a is $48 - b^2 = 48 - 16 = 32$.

4. We have P meters of fencing and want to create three adjacent rectangular enclosures as shown on the figure shown. What is the maximal area we can enclose this way?



Solution: Let x denote the small horizontal sides, and y the vertical sides.

$$\begin{aligned} P &= 6x + 4y \implies y = \frac{1}{4}(P - 6x) = -\frac{3}{2}x + \frac{P}{4} \\ A &= 3xy = 3x\left(-\frac{3}{2}x + \frac{P}{4}\right) = -\frac{9}{2}x^2 + \frac{3P}{4}x \end{aligned}$$

We compute the domain of this function. Both x and y must be positive. This gives us the following inequalities: $x > 0$ and $y = -\frac{3}{2}x + \frac{P}{4} > 0$

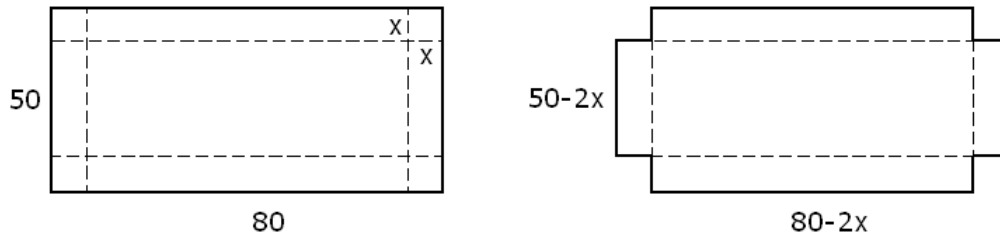
$$\begin{aligned} -\frac{3}{2}x + \frac{P}{4} &> 0 \quad \text{solve for } x \\ \frac{P}{4} &> \frac{3}{2}x \\ \frac{P}{6} &> x \implies \text{domain: } \left(0, \frac{P}{6}\right) \end{aligned}$$

The area function $A(x) = -\frac{9}{2}x^2 + \frac{3P}{4}x$ is a downward turning parabola, so its vertex is its maximum. We can find it by completing the square or by differentiation:

$$A(x) = -\frac{9}{2}x^2 + \frac{3P}{4}x = -\frac{9}{2}\left(x^2 - \frac{P}{6}x\right) = -\frac{9}{2}\left(\left(x - \frac{P}{12}\right)^2 - \frac{P^2}{144}\right) = -\frac{9}{2}\left(x - \frac{P}{12}\right)^2 + \frac{P^2}{32}$$

When $x = \frac{P}{12}$, then the maximal area is $\frac{P^2}{32}$.

5. A rectangular box, open at the top, is to be constructed from a rectangular sheet of cardboard 50 centimeters by 80 centimeters by cutting out equal squares in the corners and folding up the sides. What sides squares should be cut out for the container to have maximal volume?



Solution: The base of the box is a rectangle with sides $50 - 2x$ and $80 - 2x$. The height of the box is x . Thus the volume of the box, as a function of x is

$$V(x) = (80 - 2x)x(50 - 2x) = 4(x^3 - 65x^2 + 1000x)$$

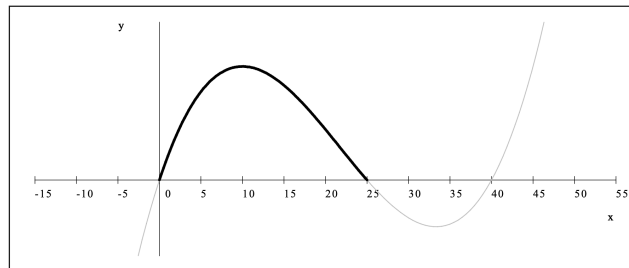
The domain is not the entire number line, clearly all sides must be positive. after we solve the inequalities $80 - 2x > 0$ and $50 - 2x > 0$ and $x > 0$, we obtain the domain $(0, 25)$. Since V is a cubic function with a positive leading coefficient, the relative extrema at the smaller x -value is a maximum, and the extrema at the greater x -value is a relative minimum. The domain is $(0, 25)$.

$$V(x) = 4(x^3 - 65x^2 + 1000x) \implies V'(x) = 4(3x^2 - 130x + 1000)$$

We factor V' to find the zeroes of the derivative

$$\begin{aligned} V'(x) &= 4(3x^2 - 130x + 1000) = 4(3x - 100)(x - 10) \\ V'(x) &= 0 \implies x_1 = 10 \quad x_2 = \frac{100}{3} \end{aligned}$$

The relative maximum is at $x = 10$, which is within the domain $(0, 25)$.



So we should cut out squares with 10 centimeters long sides. Then the maximal volume is

$$V(10) = (80 - 2 \cdot 10)(10)(50 - 2 \cdot 10) = 18\,000.$$

6. Prove that for any real numbers a and b , if $a + b = 1$, then $a^4 + b^4 \geq \frac{1}{8}$.

Solution: We will prove that the function $f(x) = x^4 + (1-x)^4$ has an absolute minimum at $x = \frac{1}{2}$.

$$f(x) = x^4 + (1-x)^4 = x^4 + (x-1)^4$$

$$f'(x) = 4x^3 + 4(x-1)^3 = 4(x^3 + (x-1)^3)$$

We factor via the sum of cubes theorem.

$$\begin{aligned} f'(x) &= 4(x+x-1)(x^2 - x(x-1) + (x-1)^2) = 4(2x-1)(x^2 - x + 1) \\ &= 8\left(x - \frac{1}{2}\right)(x^2 - x + 1) \end{aligned}$$

In case of the quadratic factor, we complete the square to see when that expression is negative.

$$8\left(x - \frac{1}{2}\right)(x^2 - x + 1) = 8\left(x - \frac{1}{2}\right)\left(\left(x - \frac{1}{2}\right)^2 + \frac{3}{4}\right)$$

We can now see that the quadratic expression is always positive. (This is always the case with the quadratic factor in the sum or difference of two cubes.) Thus f' will be negative on $\left(-\infty, \frac{1}{2}\right)$ and positive on $\left(\frac{1}{2}, \infty\right)$. Thus f is strictly decreasing on $\left(-\infty, \frac{1}{2}\right)$ and strictly increasing on $\left(\frac{1}{2}, \infty\right)$. This means that f has an **absolute** minimum at $x = \frac{1}{2}$. We compute $f\left(\frac{1}{2}\right) = \left(\frac{1}{2}\right)^4 + \left(1 - \frac{1}{2}\right)^4 = \frac{1}{8}$. (Note: this problem can be solved by elementary methods as well. One method involves simplifying $\left(\frac{1}{2} + x\right)^4 + \left(\frac{1}{2} - x\right)^4$ and completing the square.)

7. One thousand feet of fencing is to be used to surround two areas, one square and one circular. What should the size of each area be in order that the total area be
- a) as large as possible b) as small as possible?

Solution: Let us denote the radius of the circle by r . Clearly $r \geq 0$. If we denote the side of the square by s , we have an equation for the total perimeter. We can solve that for s .

$$\begin{aligned} 1000 &= 4s + 2\pi r \\ \frac{1000 - 2\pi r}{4} &= s \implies s = \frac{500 - \pi r}{2} \end{aligned}$$

We also need $s \geq 0$

$$\begin{aligned} s &\geq 0 & 500 &\geq \pi r \\ \frac{500 - \pi r}{2} &\geq 0 & \frac{500}{\pi} &\geq r \\ 500 - \pi r &\geq 0 \end{aligned}$$

Thus the area-function, (yet to be set up) will have domain $\left[0, \frac{500}{\pi}\right]$. The endpoints mean that all 1000 feet was used for the square (in case of $r = 0$) or for the circle (in case of $r = \frac{500}{\pi}$). The area, as a function of r is now

$$A(r) = \pi r^2 + s^2 = \pi r^2 + \left(\frac{500 - \pi r}{2}\right)^2 = \left(\pi + \frac{1}{4}\pi^2\right)r^2 - 250\pi r + 62\,500$$

Since this is a quadratic expression with a positive leading coefficient, we know that the vertex is a minimum, both relative and absolute. The vertex can be found by either elementary methods or by finding the zero of $A'(r)$. Either way, we will get that $r_{\min} = \frac{500}{4 + \pi}$. This means that the circle will have radius $r = \frac{500}{4 + \pi}$ and the square will have sides

$$s_{\min} = \frac{500 - \pi r_{\min}}{2} = \frac{1000}{\pi + 4} \text{ since}$$

$$\begin{aligned} s_{\min} &= \frac{500 - \pi r_{\min}}{2} = \frac{500 - \pi \left(\frac{500}{4 + \pi} \right)}{2} = \frac{500 - \pi \left(\frac{500}{4 + \pi} \right)}{2} \cdot \frac{4 + \pi}{4 + \pi} = \frac{500(4 + \pi) - 500\pi}{2(4 + \pi)} \\ &= \frac{2000 + 500\pi - 500\pi}{2(4 + \pi)} = \frac{2000}{2(4 + \pi)} = \frac{1000}{\pi + 4} = 2r_{\min} \end{aligned}$$

These dimensions give us the minimal area

$$\begin{aligned} A_{\min} &= \pi r_{\min}^2 + s_{\min}^2 = \pi \left(\frac{500}{4 + \pi} \right)^2 + \left(\frac{1000}{\pi + 4} \right)^2 = \frac{500^2 \pi + 1000^2}{(4 + \pi)^2} = \frac{500^2 \pi + (2 \cdot 500)^2}{(4 + \pi)^2} \\ &= \frac{500^2 \pi + 4 \cdot 500^2}{(4 + \pi)^2} = \frac{500^2 (\pi + 4)}{(4 + \pi)^2} = \frac{500^2}{(4 + \pi)} = \frac{250\,000}{\pi + 4} \end{aligned}$$

But what about the maximum? A quadratic expression does not have a relative maximum if its leading coefficient is positive. The absolute maximum must be at one of the end-points of the interval $\left[0, \frac{500}{\pi}\right]$. We need to compare these areas. When $r = 0$, all thousand feet was used for the square. The area is then

$$A(0) = 250^2$$

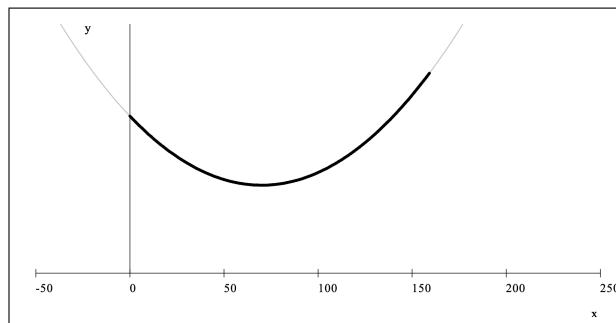
or, using our expression for the area,

$$A(r) = \left(\pi + \frac{1}{4}\pi^2 \right) r^2 - 250\pi r + 62\,500 \implies A(0) = 62\,500$$

When $r = \frac{500}{\pi}$, all thousand feet was used for the circle. Then the area is

$$A\left(\frac{500}{\pi}\right) = \pi r^2 = \pi \left(\frac{500}{\pi}\right)^2 = \frac{500^2}{\pi}$$

Since $\frac{500^2}{\pi} \approx 79\,577.471\,545\,5$, $A(0) < A\left(\frac{500}{\pi}\right)$



which means that the greatest area occurs if we use all thousand feet for the circle.

8. A Norman window has the outline of a semicircle on top of a rectangle, as shown on the picture below. Find the dimensions of the window that can be built using 8 meters of wood and has the maximal area.

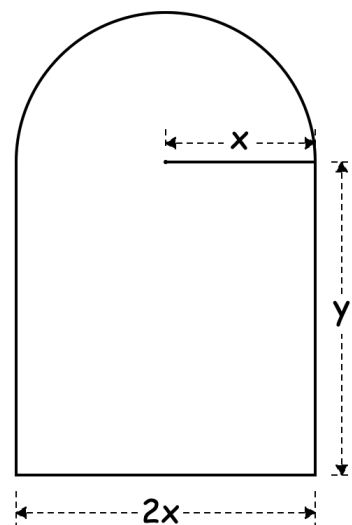
Solution: Let x denote half of the bottom side. (Also the radius.)

Let y denote the vertical side. Then $2x + 2y + \pi x = 8$. We solve for y and obtain

$$y = -\left(\frac{\pi + 2}{2}\right)x + 4$$

The domain will be determined by the conditions $x > 0$ and $y \geq 0$. The domain is $\left(0, \frac{8}{\pi + 2}\right]$. We set up the area-function:

$$A(x) = 2xy + \frac{\pi x^2}{2} = 2x\left(-\left(\frac{\pi + 2}{2}\right)x + 4\right) + \frac{\pi x^2}{2} = \left(-\frac{\pi}{2} - 2\right)x^2 + 8x$$



The vertex of this downward opening parabola is at $x = \frac{8}{\pi + 4}$, which is in the domain. Then $y = \frac{8}{\pi + 4}$. The dimensions are: $2x = \frac{16}{\pi + 4}$ wide and $x + y = \frac{16}{\pi + 4}$ tall.

9. Find the point(s) on the arc of the parabola $y = x^2$ that are nearest to the point $(0, 2)$.

Solution: Let $P(x, x^2)$ be a point on the graph. The distance between P and $(0, 2)$ is

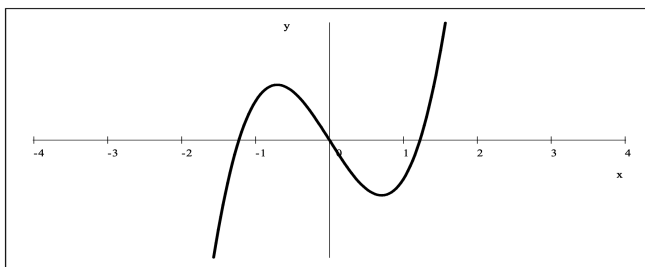
$$d = \sqrt{(x - 0)^2 + (x^2 - 2)^2} = \sqrt{x^2 + x^4 - 4x^2 + 4} = \sqrt{x^4 - 3x^2 + 4}$$

It is a common trick to find the minimum of d^2 instead of d . Let $f(x)$ denote the square of the distance between the point (x, x^2) and $(0, 2)$.

$$f(x) = x^4 - 3x^2 + 4$$

We differentiate: $f'(x) = 4x^3 - 6x = 4x\left(x^2 - \frac{3}{2}\right) = 4x\left(x + \sqrt{\frac{3}{2}}\right)\left(x - \sqrt{\frac{3}{2}}\right)$

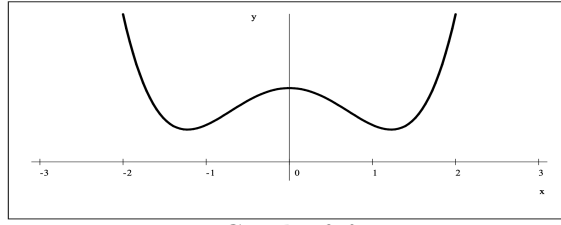
Since f' is a cubic function with a positive leading coefficient, and x -intercepts at $x = -\sqrt{\frac{3}{2}}$ and $x = \sqrt{\frac{3}{2}}$. We sketch the graph of f' .



Based on the signs of f' , we determine that f has a relative minimum at $x = -\sqrt{\frac{3}{2}}$ and $x = \sqrt{\frac{3}{2}}$ and a relative maximum at $x = 0$. We now evaluate f at $x = -\sqrt{\frac{3}{2}}, 0, \sqrt{\frac{3}{2}}$.

$$f(0) = 4, \quad f\left(-\sqrt{\frac{3}{2}}\right) = \frac{7}{4}, \quad f\left(\sqrt{\frac{3}{2}}\right) = \frac{7}{4}$$

We can now plot the graph of f .



The absolute minimum is at $x = -\sqrt{\frac{3}{2}}$ and $x = \sqrt{\frac{3}{2}}$ indicating that the points closest to $(0, 2)$ are $\left(-\sqrt{\frac{3}{2}}, \frac{3}{2}\right)$ and $\left(\sqrt{\frac{3}{2}}, \frac{3}{2}\right)$.

10. A company wants to manufacture cylindrical aluminum cans using 100 cm^2 of aluminum. What dimensions would guarantee the maximal volume?

Solution: Let h denote the height of the can, and r denote the radius of the base circle. The surface of a cylinder is $A = 2\pi r^2 + 2\pi r h$. We solve for h .

$$100 = 2\pi r^2 + 2\pi r h \implies h = \frac{100 - 2\pi r^2}{2\pi r} = -r + \frac{50}{\pi r}$$

The volume of a cylinder is $V = \pi r^2 h$. We substitute $h = \frac{100 - 2\pi r^2}{2\pi r}$ into this equation and obtain V as a function of r .

$$\begin{aligned} V(r) &= \pi r^2 \left(\frac{100 - 2\pi r^2}{2\pi r} \right) = r \left(\frac{100 - 2\pi r^2}{2} \right) = r(50 - \pi r^2) \\ V(r) &= -\pi r^3 + 50r \end{aligned}$$

The volume is a cubic function with a negative leading coefficient. We can also determine the domain of V by stating that both r and h must be positive.

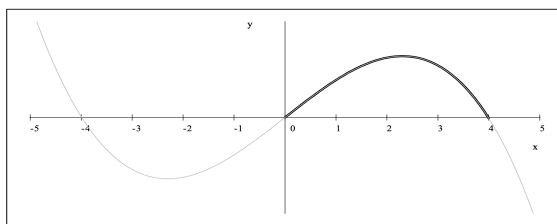
$$\begin{aligned} h &> 0 \text{ and } r > 0 \\ \frac{100 - 2\pi r^2}{2\pi r} &> 0 && \text{multiply by } 2\pi r > 0 \\ 100 - 2\pi r^2 &> 0 \\ 100 &> 2\pi r^2 \\ \frac{50}{\pi} &> r^2 \implies -\sqrt{\frac{50}{\pi}} < r < \sqrt{\frac{50}{\pi}} \quad \text{and } r > 0 \end{aligned}$$

The domain of the volume function is thus $\left(0, \sqrt{\frac{50}{\pi}}\right)$.

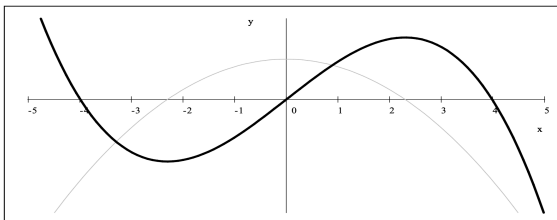
We differentiate $V(r)$.

$$V'(r) = -3\pi r^2 + 50 = -3\pi \left(r^2 - \frac{50}{3\pi} \right) = -3\pi \left(r + \sqrt{\frac{50}{3\pi}} \right) \left(r - \sqrt{\frac{50}{3\pi}} \right)$$

The derivative is a downward turned parabola with x -intercepts at $-\sqrt{\frac{50}{3\pi}}$ and $\sqrt{\frac{50}{3\pi}}$. The positive one, $r = \sqrt{\frac{50}{3\pi}}$ is where the volume has a relative maximum. We can now sketch the volume function V .



If we restrict the function to its domain, $\left(0, \sqrt{\frac{50}{\pi}}\right)$, then it is clear that the relative maximum we found is also an absolute maximum.



So the maximal volume will be obtained with $r = \sqrt{\frac{50}{3\pi}}$. Then $h = -r + \frac{50}{\pi r}$

$$\begin{aligned} h &= -\sqrt{\frac{50}{3\pi}} + \frac{50}{\pi \sqrt{\frac{50}{3\pi}}} = -\sqrt{\frac{50}{3\pi}} + \frac{50}{\pi} \sqrt{\frac{3\pi}{50}} = -\sqrt{\frac{50}{3\pi}} + \sqrt{\frac{50^2}{\pi^2} \cdot \frac{3\pi}{50}} = \\ &= -\sqrt{\frac{50}{3\pi}} + \sqrt{9 \cdot \frac{50}{3\pi}} = -\sqrt{\frac{50}{3\pi}} + \sqrt{9} \sqrt{\frac{50}{3\pi}} = -\sqrt{\frac{50}{3\pi}} + 3\sqrt{\frac{50}{3\pi}} = 2\sqrt{\frac{50}{3\pi}} = 2r \end{aligned}$$

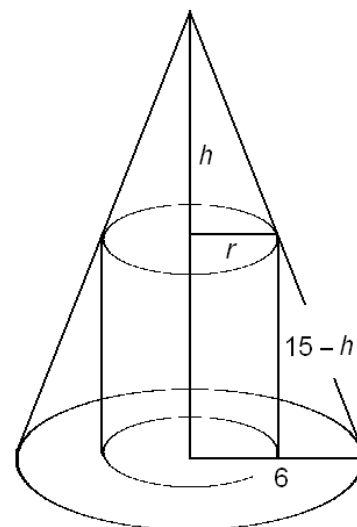
and so the maximal volume is $V\left(\sqrt{\frac{50}{3\pi}}\right)$.

11. We write a cylinder into a cone as shown on the picture. The cone is of height 15cm and its base has a radius of 6cm. What dimensions of the cylinder would guarantee the greatest volume? What is the maximal volume?

Solution: The cone above the cylinder and the big cone are similar to each other. Let us denote the height of this cone by h and the radius of its base by r . Then the cylinder's height is $15 - h$ and its base has radius r .

Since the two cones are similar, $\frac{15}{6} = \frac{h}{r}$. We solve for h and obtain $h = \frac{5}{2}r$. We can now express the volume of the cylinder as a function of r .

$$V_{\text{cylinder}} = \pi r^2 (15 - h) = \pi r^2 \left(15 - \frac{5}{2}r\right) = \pi \left(-\frac{5}{2}r^3 + 15r^2\right)$$

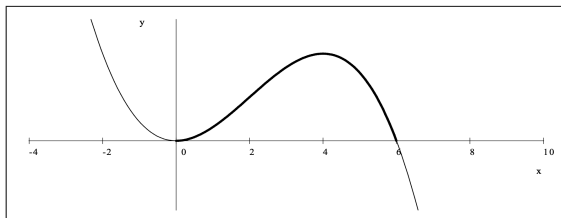


It is also clear that $0 < r < 6$ and so the volume of the cylinder as a function of r is

$$V(r) = \pi \left(-\frac{5}{2}r^3 + 15r^2\right) \text{ on domain } (0, 6)$$

$V(r)$ is clearly a cubic function with a negative leading coefficient and zeroes at 0 and r , since

$$V(r) = \pi r^2 \left(15 - \frac{5}{2}r\right) = \pi r^2 \left(-\frac{5}{2}\right)(r - 6) = -\frac{5}{2}\pi r^2(r - 6)$$



We can see from the graph that there is a relative minimum at 0 and a relative maximum between 0 and 6 that is an absolute maximum if the domain is $(0, 6)$. To find this maximum, we differentiate $V(r)$.

$$\begin{aligned} V(r) &= \pi \left(-\frac{5}{2}r^3 + 15r^2\right) \\ V'(r) &= \pi \left(-\frac{15}{2}r^2 + 30r\right) = -\frac{15}{2}\pi r(r - 4) \end{aligned}$$

The derivative has zeroes at $r = 0$ and $r = 4$. Thus the maximum volume is achieved when $r = 4$. Then clearly $h = \frac{5}{2}r = \frac{5}{2} \cdot 4 = 10$. This is not the height of the cylinder; that is $15 - h = 15 - 10 = 5$. Dimensions are in centimeters and so the maximal volume is $V_{\max} = \pi(4\text{cm})^2(5\text{cm}) = 80\pi\text{cm}^3$ when the radius of the cylinder is 4cm and its height is 5cm.

12. Consider a sphere with radius R and all right cones we can write into it. Which one has the greatest volume?

Solution: Let us denote the height of the cone by h and the base of its radius by r . Consider now the picture shown below.

We can write the Pythagorean Theorem on the right triangle and get

$$R^2 = (h - R)^2 + r^2 \text{ from which we solve for } r^2 \text{ and get that}$$

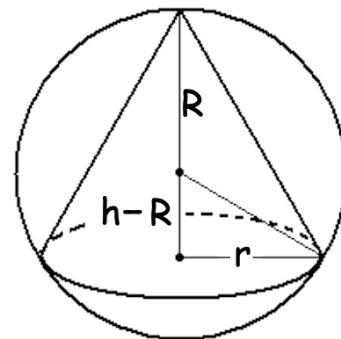
$$r^2 = R^2 - (h - R)^2 = R^2 - (R^2 + h^2 - 2hR) = 2hR - h^2$$

Now the volume of the cone is $V = \frac{1}{3}\pi r^2 h$. We substitute $r^2 = 2hR - h^2$ into this and the volume becomes a function of h .

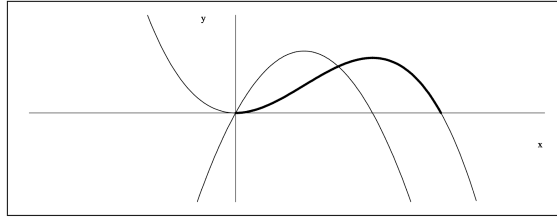
$$V(h) = \frac{1}{3}\pi r^2 h = \frac{1}{3}\pi (2hR - h^2) h = \frac{\pi}{3}(-h^3 + 2h^2 R)$$

This is a cubic function with a negative leading coefficient with a domain $(R, 2R)$. We differentiate V and factor V' to find its zeroes.

$$V'(h) = \frac{\pi}{3}(-3h^2 + 4hR) = \frac{\pi}{3}(-3)\left(h^2 - \frac{4hR}{3}\right) = -\pi h\left(h - \frac{4R}{3}\right)$$



V' is a downward opening parabola with zeroes at $h = 0$ and $h = \frac{4R}{3}$.



The greater one represents a relative maximum for f . On the restricted domain $(R, 2R)$, V has an absolute maximum at $h = \frac{4}{3}R$ so it is our solution. Then

$$r = \sqrt{2hR - h^2} = \sqrt{2\left(\frac{4}{3}R\right)R - \left(\frac{4}{3}R\right)^2} = \sqrt{\frac{8}{3}R^2 - \frac{16}{9}R^2} = R\sqrt{\frac{8}{3} - \frac{16}{9}} = R\sqrt{\frac{8}{9}} = \frac{2\sqrt{2}}{3}R$$

and

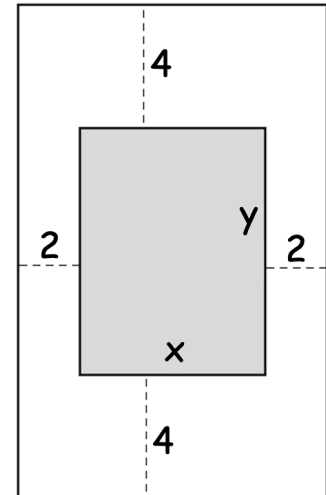
$$V_{\max} = \frac{1}{3}\pi r^2 h = \frac{1}{3}\pi \left(\sqrt{\frac{8}{9}R}\right)^2 \left(\frac{4}{3}R\right) = \frac{\pi}{3} \cdot \frac{8}{9} \cdot \frac{4}{3}R^3 = \frac{32}{81}\pi R^3$$

13. We are designing a poster to contain 50in^2 of printing with a 4-inch wide margin at the top and bottom and a 2-inch wide margin at each side. What overall dimensions will minimize the amount of paper used?

Solution: Let x and y denote the vertical and horizontal sides of the printed area. Then the printed area is $A_{\text{pr}} = xy = 50$. We solve for y in terms of x and get $y = \frac{50}{x}$. The dimensions of the entire paper is

$$A = (x + 8)(y + 4) = (x + 8)\left(\frac{50}{x} + 4\right)$$

This is a function of x , and we are looking for the minimum of this expression.



$$A(x) = (x + 8)\left(\frac{50}{x} + 4\right) = 50 + 4x + 8 \cdot \frac{50}{x} + 32 = 82 + 4x + \frac{400}{x} = 82 + 4x + 400x^{-1}$$

$$A'(x) = 4 - 400x^{-2} = 4 - \frac{400}{x^2} = \frac{4x^2 - 400}{x^2} = \frac{4(x^2 - 100)}{x^2} = \frac{4(x - 10)(x + 10)}{x^2}$$

Let us look first at the numerator. $y = 4(x - 10)(x + 10)$ is an upward opening parabola with x -intercepts at $x = -10$ and 10 . This indicates a relative maximum at $x = -10$ and $x = 10$. How does the denominator x^2 affect the behavior of the function?

First of all, x^2 is never negative and so it does not have any effect on the sign of A' . But x^2 is zero at $x = 0$ which causes the function A' to be undefined at $x = 0$. However, our domain does not include zero. On our domain, $(0, \infty)$, the denominator is always positive and the only critical number is $x = 10$. There A' changes sign from negative to positive indicating a relative minimum.

So the optimal dimensions are associated with $x = 10$. Then $y = \frac{50}{x} = \frac{50}{10} = 5$ and so that paper should be $(x + 8)$ by $(y + 4)$ i.e. 18 inches by 9 inches.

We should also make sure that the relative minimum we found is also an absolute minimum. The fact that A' is negative on $(0, 10)$ and positive on $(10, \infty)$ indicate that A is decreasing on $(0, 10)$ and increasing on $(10, \infty)$. That implies that A has an absolute minimum on $(0, \infty)$.

14. Let a and b be positive numbers such that $ab = 10$. Find the lowest value of $a^2 + 4b^2$.

Solution: We solve for a in terms of b : $a = \frac{10}{b}$. Then the expression $a^2 + 4b^2$ becomes

$$P(b) = \left(\frac{10}{b}\right)^2 + 4b^2 = 4b^2 + \frac{100}{b^2} = 4b^2 + 100b^{-2}$$

We differentiate this:

$$\begin{aligned} P'(b) &= 8b + 100(-2)b^{-3} = 8b - \frac{200}{b^3} = \frac{8b^4 - 200}{b^3} \\ &= \frac{8(b^4 - 25)}{b^3} = \frac{8(b^2 + 5)(b^2 - 5)}{b^3} = \frac{8(b^2 + 5)(b + \sqrt{5})(b - \sqrt{5})}{b^3} \end{aligned}$$

The critical numbers for P are $-\sqrt{5}$, 0 , and $\sqrt{5}$. All relative maximums or minimums will be here. We can figure out when P' is positive and negative by sorting out the signs of each factor in the numerator and denominator.

	$b < -\sqrt{5}$	$-\sqrt{5} < b < 0$	$0 < b < \sqrt{5}$	$b > \sqrt{5}$
$(b^2 + 5)$	+	+	+	+
$(b + \sqrt{5})$	-	+	+	+
$(b - \sqrt{5})$	-	-	-	+
b^3	-	-	+	+
P'	-	+	-	+

Based on the signs of P' only, P has a relative minimum at $b = -\sqrt{5}$ and $\sqrt{5}$ and a relative maximum at 0 . However, the function does not have a relative maximum at zero. Looking at the formula for the original function, $P(b) = 4b^2 + \frac{100}{b^2}$, we see that there is a vertical asymptote and the graph shoots up toward plus infinity on both sides of the asymptote. Not to mention the fact that a and b must both be positive. Since b must be positive, we may consider P on the domain $(0, \infty)$. On this domain, P is continuous and differentiable everywhere, is decreasing on $(0, \sqrt{5})$ and increasing on $(\sqrt{5}, \infty)$ and so P has an absolute minimum at $b = \sqrt{5}$.

If $b = \sqrt{5}$, then

$$P(\sqrt{5}) = \left(\frac{10}{\sqrt{5}}\right)^2 + 4(\sqrt{5})^2 = \frac{100}{5} + 4 \cdot 5 = 20 + 20 = 40$$

Thus the smallest possible value of $a^2 + 4b^2$ is 40.

Note: It is becoming more and more laborious to sort out when the derivative is positive and negative. Soon enough there will be a point where this becomes difficult. In such cases, we can usually use the **second derivative test**.

The second derivative test states that if a function f is twice differentiable at a number a and

- 1) If $f'(a) = 0$ and $f''(a) > 0$ then f has a relative minimum at $x = a$.
- 2) If $f'(a) = 0$ and $f''(a) < 0$ then f has a relative maximum at $x = a$.
- 3) If $f'(a) = 0$ and $f''(a) = 0$, then the second derivative test did not yield useful information on the behavior of f at $x = a$.

We can apply the second derivative test in this problem. $P'(\sqrt{5}) = 0$. We first compute the second derivative of P

$$\begin{aligned} P(b) &= 4b^2 + 100b^{-2} \\ P'(b) &= 8b - 200b^{-3} \\ P''(b) &= 8 - 200(-3)b^{-4} = 8 + \frac{600}{b^4} \end{aligned}$$

We evaluate the second derivative at $\sqrt{5}$ (although it already is obvious that $P''(\sqrt{5})$ is positive, given the formula for P'')

$$P''(\sqrt{5}) = 8 + \frac{600}{(\sqrt{5})^4} = 8 + \frac{600}{25} = 32$$

Since $P'(\sqrt{5}) = 0$ and $P''(\sqrt{5}) > 0$, the function P has a relative minimum at $b = \sqrt{5}$.

15. A closed box with a square base is to have a volume of 250 cubic meters. The material for the top and bottom of the box costs \$2 per square meter, and the material for the sides costs \$1 per square meter. Can the box be constructed for less than \$300?

Solution: Let x denote the side of the square base, and h denote the height of the box. Then $V = hx^2$ gives us

$$\begin{aligned} hx^2 &= 250 \\ h &= \frac{250}{x^2} \end{aligned}$$

We now set up the cost function, $C(x)$. The top and bottom each cost \$2 per square meter, and have area x^2 . The four sides each have area $xh = x \left(\frac{250}{x^2} \right) = \frac{250}{x}$ and cost \$1 per square meter. Thus

$$C(x) = 2 \cdot 2 \cdot x^2 + 4 \cdot 1 \cdot \frac{250}{x} = 4x^2 + \frac{1000}{x} = 4x^2 + 1000x^{-1}$$

We are looking for the maximum of $C(x)$. We will differentiate C first.

$$C'(x) = 8x + 1000(-1)x^{-2} = 8x - \frac{1000}{x^2} = \frac{8x^3 - 1000}{x^2} = \frac{8(x^3 - 125)}{x^2}$$

The numerator can be factored via the difference of cubes theorem, which states that

$$A^3 - B^3 = (A - B)(A^2 + AB + B^2)$$

where the second factor can not be factored further as it is a sum of two squares. We can see that by completing the square

$$A^2 + AB + B^2 = \underbrace{A^2 + AB + \frac{B^2}{4}}_{\left(A + \frac{B}{2}\right)^2} - \frac{B^2}{4} + B^2 = \left(A + \frac{B}{2}\right)^2 + \frac{3}{4}B^2$$

So we can factor the numerator in C' via the difference of cubes theorem

$$C'(x) = \frac{8(x^3 - 125)}{x^2} = \frac{8(x - 5)(x^2 + 5x + 25)}{x^2}$$

Recall that the domain is $(0, \infty)$. On this domain, C' is defined everywhere and has only one zero, at $x = 5$. Since both x^2 and $x^2 + 5x + 25$ are positive for all values of x , C' will change sign from negative to positive at $x = 5$, indicating a minimum of C . Thus, the lowest possible cost will be associated with $x = 5$. The actual cost is then

$$C(5) = 4 \cdot 5^2 + \frac{1000}{5} = 300$$

Thus, we can not construct this box for less than \$300.

Please note that instead of factoring the difference of cubes, we can apply the second derivative test.

$$\begin{aligned} C(x) &= 4x^2 + 1000x^{-1} \\ C'(x) &= 8x - 1000x^{-2} \\ C''(x) &= 8 - 1000(-2)x^{-3} = 8 + \frac{2000}{x^3} \end{aligned}$$

Again, the formula for C'' indicates positive values on all positive numbers, so $C''(5)$ is positive. However, in case we didn't notice that,

$$C''(5) = 8 + \frac{2000}{5^3} = 24$$

and so $C'(5) = 0$ and $C''(5) > 0$ indicates a relative minimum at $x = 5$. The advantage of factoring the sum of cubes is that it is much easier to argue that the minimum is an absolute minimum.

16. A company wants to manufacture cylindrical aluminum cans with a volume of 1000 cubic centimeters (one liter). What dimensions would guarantee the minimal amount of aluminum needed to produce a can?

Solution: Let h denote the height of the can, and r denote the radius of the base circle.

$$\pi r^2 h = 1000 \quad h = \frac{1000}{\pi r^2}$$

The domain is $(0, \infty)$

$$S(r) = 2\pi r h + 2\pi r^2 = 2\pi r \left(\frac{1000}{\pi r^2} \right) + 2\pi r^2 = 2\pi r^2 + \frac{2000}{r}$$

$$S'(r) = 4\pi r - \frac{2000}{r^2} = \frac{4\pi r^3 - 2000}{r^2}$$

$$4\pi r - \frac{2000}{r^2} = 0$$

$$4\pi r = \frac{2000}{r^2}$$

$$\pi r^3 = 500 \implies r = \sqrt[3]{\frac{500}{\pi}} \approx 5.41926$$

and

$$h = \frac{1000}{\pi \left(\sqrt[3]{\frac{500}{\pi}} \right)^2} = \frac{1000}{\pi (500^{2/3}) (\pi^{-2/3})} = \frac{2 \cdot 500}{(500^{2/3}) (\pi^{1/3})} = \frac{2 \cdot 500^{1/3}}{(\pi^{1/3})} = 2 \sqrt[3]{\frac{500}{\pi}} = 2r \approx 10.838521$$

But is this a minimum we found? We can still factor S' (and we should see that the minimum is an absolute one)

$$\begin{aligned} S'(r) &= 4\pi r - \frac{2000}{r^2} = \frac{4\pi r^3 - 2000}{r^2} = \frac{4\pi \left(r^3 - \frac{2000}{4\pi} \right)}{r^2} = \frac{4\pi \left(r^3 - \frac{500}{\pi} \right)}{r^2} \\ &= \frac{4\pi \left(r - \sqrt[3]{\frac{500}{\pi}} \right) \left(r^2 + r \left(\sqrt[3]{\frac{500}{\pi}} \right) + \left(\sqrt[3]{\frac{500}{\pi}} \right)^2 \right)}{r^2} \end{aligned}$$

The last, long factor in the numerator is always positive, and so is the denominator. So, the signs of S' are determined by the linear factor: S' is negative before $\sqrt[3]{\frac{500}{\pi}}$ and positive after $\sqrt[3]{\frac{500}{\pi}}$. This implies that S is decreasing before $\sqrt[3]{\frac{500}{\pi}}$ and increasing after, and so we indeed found a minimum, in fact an absolute minimum at $r = \sqrt[3]{\frac{500}{\pi}}$.

This can also be done using the second derivative: $S''(r) = 4\pi + \frac{4000}{r^3}$

Since S'' is positive on the entire domain (recall $r > 0$), S' is strictly increasing on its entire domain. This means that S' is negative before its only zero at $\sqrt[3]{\frac{500}{\pi}}$ and positive after. This implies that S is decreasing before $\sqrt[3]{\frac{500}{\pi}}$ and increasing after, and so we indeed found an absolute minimum.

17. A company determines that if n is the number of items produced, they can all be sold at a price of $p(n) = \sqrt{1200 - 0.2n}$. What is the greatest revenue possible?

Solution: If we sell all n products, the revenue is $R(n) = n\sqrt{1200 - 0.2n}$. Since the price should be positive, the domain of this function is determined by $n > 0$ and $1200 - 0.2n > 0$. We solve these inequalities and obtain the domain $(0, 6000)$. We will find the maximum of this function by finding the zeroes of the derivative.

$$\begin{aligned} R'(n) &= \sqrt{1200 - 0.2n} + n \frac{1}{2\sqrt{1200 - 0.2n}} (-0.2) = \sqrt{1200 - 0.2n} - \frac{0.1n}{\sqrt{1200 - 0.2n}} \\ &= \frac{1200 - 0.2n}{\sqrt{1200 - 0.2n}} - \frac{0.1n}{\sqrt{1200 - 0.2n}} = \frac{1200 - 0.2n - 0.1n}{\sqrt{1200 - 0.2n}} = \frac{-0.3n + 1200}{\sqrt{1200 - 0.2n}} = \frac{-0.3(n - 4000)}{\sqrt{1200 - 0.2n}} \end{aligned}$$

The derivative has only one zero, at $n = 4000$. The denominator is positive for all n with $0 < n < 6000$ and the numerator is positive before 4000 and negative after 4000. Consequently, $R(n)$ has a relative and absolute maximum at $n = 4000$. The maximal possible revenue is then $R(4000)$.

$$R(4000) = 4000\sqrt{1200 - 0.2(4000)} = 4000\sqrt{1200 - 800} = 4000\sqrt{400} = 4000(20) = 80\,000$$

18. A company has \$120 000 to spend on the development and promotion of a new product. The company estimates that if x is spent on the development and y is spent on promotion, then approximately $\frac{x^{1/2}y^{3/2}}{400\,000}$ items of new product will be sold. Based on this estimate, what is the maximum number of products that the company can sell?

Solution: Since $y = 120\,000 - x$, we can write the number sold as a function of x .

$$N(x) = \frac{1}{400\,000} x^{1/2} (120\,000 - x)^{3/2}$$

The domain is clearly $[0, 120\,000]$. We differentiate N .

$$\begin{aligned} N'(x) &= \frac{1}{400\,000} \left[\frac{1}{2} x^{-1/2} (120\,000 - x)^{3/2} + x^{1/2} \left(\frac{3}{2} \right) (120\,000 - x)^{1/2} (-1) \right] \\ &= \frac{1}{400\,000} \left(\frac{1}{2} \right) \left[\frac{(120\,000 - x)^{3/2}}{\sqrt{x}} - 3\sqrt{x} (120\,000 - x)^{1/2} \right] \\ N'(x) &= \frac{1}{800\,000} \left(\frac{\sqrt{120\,000 - x} (120\,000 - x)}{\sqrt{x}} - 3\sqrt{x} \sqrt{120\,000 - x} \right) \quad \text{factor out } \sqrt{120\,000 - x} \\ &= \frac{\sqrt{120\,000 - x}}{800\,000} \left(\frac{(120\,000 - x)}{\sqrt{x}} - 3\sqrt{x} \right) \quad \text{bring difference to common denominator} \\ &= \frac{\sqrt{120\,000 - x}}{800\,000} \left(\frac{(120\,000 - x)}{\sqrt{x}} - 3\frac{x}{\sqrt{x}} \right) = \frac{\sqrt{120\,000 - x}}{800\,000} \left(\frac{120\,000 - x - 3x}{\sqrt{x}} \right) \\ &= \frac{\sqrt{120\,000 - x}}{800\,000} \left(\frac{120\,000 - 4x}{\sqrt{x}} \right) = \frac{-4\sqrt{120\,000 - x} (x - 30\,000)}{800\,000\sqrt{x}} \\ &= \frac{\sqrt{120\,000 - x}}{800\,000\sqrt{x}} [-4(x - 30\,000)] \\ N'(x) &= \frac{\sqrt{120\,000 - x}}{800\,000\sqrt{x}} (-4)(x - 30\,000) \end{aligned}$$

The first factor, $\frac{\sqrt{120\,000 - x}}{800\,000\sqrt{x}}$ is positive for all x in the domain, except for the endpoints. The second factor, $-4(x - 30\,000)$ is positive before 30 000, zero at $x = 30\,000$, and negative after 30 000, indicating an absolute maximum at $x = 30\,000$. Then $y = 120\,000 - 30\,000 = 90\,000$ and the maximal number sold is then

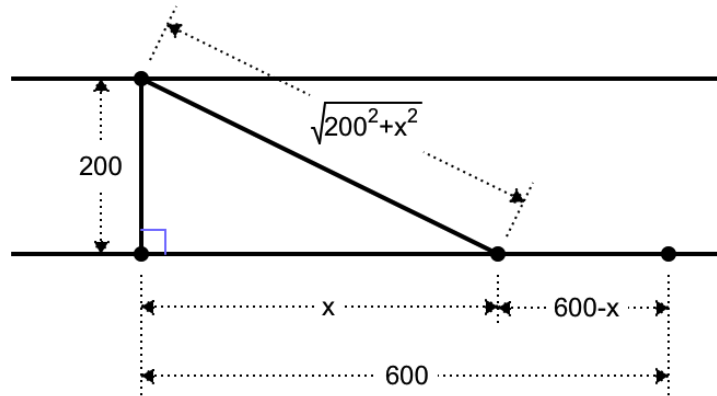
$$N(30\,000) = \frac{x^{1/2}y^{3/2}}{400\,000} = \frac{30\,000^{1/2}90\,000^{3/2}}{400\,000} \approx 11\,691.342\,95$$

The integer nearest to this number is 11 691.

19. An underground telephone cable is to be laid between two boat docks on opposite banks of a straight river. One boathouse is 600 meters downstream from the other. The river is 200 meters wide. If the cost of laying the cable is \$50 per meter under water and \$30 per meter on land, how should the cable to be laid to minimize cost?

a) Find the lowest cost possible.

Solution: Let us denote by x - as shown on the picture below - the distance alongside the river of the part of the cable to be laid under the water.



Then the length of the cable laid under water is $\sqrt{200^2 + x^2}$ and $600 - x$ on the ground. We can now express the cost as a function of x .

$$C(x) = 50\sqrt{200^2 + x^2} + 30(600 - x) \quad \text{on domain } [0, 600]$$

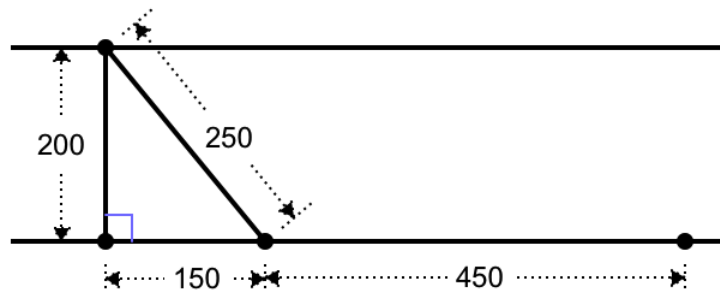
We find the minimum of C by differentiating C .

$$C'(x) = 50 \frac{1}{2\sqrt{200^2 + x^2}} (2x) + 30(-1) = \frac{50x}{\sqrt{200^2 + x^2}} - 30$$

To find the extrema, we solve for the zeroes of the derivative.

$$\begin{aligned} C'(x) &= 0 & 25x^2 &= 9(x^2 + 40\,000) \\ \frac{50x}{\sqrt{200^2 + x^2}} - 30 &= 0 & 25x^2 &= 9x^2 + 360\,000 \\ \frac{50x}{\sqrt{200^2 + x^2}} &= 30 & 16x^2 &= 360\,000 \\ 50x &= 30\sqrt{200^2 + x^2} & x^2 &= 22\,500 \\ 5x &= 3\sqrt{200^2 + x^2} & x &= \pm 150 \\ 25x^2 &= 9(200^2 + x^2) \end{aligned}$$

Since the domain is $[0, 600]$, the only possibility is $x = 150$. Thus the minimum cost is $C(150)$.



$$C(150) = 50\sqrt{200^2 + 150^2} + 30(600 - 150) = 50 \cdot 250 + 30 \cdot 450 = \$26\,000$$

b) Use the second derivative test to prove that we found a relative minimum in part a).

Solution: We already know that $C'(150) = 0$. If $C''(150)$ is positive, then C has a relative minimum at $x = 150$. If $C''(150)$ is negative, then C has a relative maximum at $x = 150$. If $C'' = 0$, the second derivative test is inconclusive.

$$\begin{aligned} C'(x) &= \frac{50x}{\sqrt{200^2 + x^2}} - 30 \\ C''(x) &= \frac{50\sqrt{200^2 + x^2} - 50x \frac{1}{2\sqrt{200^2 + x^2}}(2x)}{200^2 + x^2} = \frac{50\sqrt{200^2 + x^2} - \frac{50x^2}{\sqrt{200^2 + x^2}}}{200^2 + x^2} \\ C'''(x) &= \frac{50}{200^2 + x^2} \left(\sqrt{200^2 + x^2} - \frac{x^2}{\sqrt{200^2 + x^2}} \right) \\ C'''(150) &= \frac{50}{200^2 + 150^2} \left(\sqrt{200^2 + 150^2} - \frac{150^2}{\sqrt{200^2 + 150^2}} \right) = \frac{16}{125} > 0 \end{aligned}$$

Since its second derivative is positive at $x = 150$, the function C has a relative minimum at $x = 150$.

c) Prove that the relative minimum we found in part a) is also an absolute minimum on the domain $[0, 600]$.

Solution: Since the function $C(x)$ is continuous on the closed interval $[0, 600]$, it achieves the absolute minimum and absolute maximum. To find these, we only need to evaluate the function at the relative extrema and the endpoints of the interval. We need to compute $C(0)$, $C(150)$, and $C(600)$. These values are $C(0) = \$28\,000$, $C(150) = \$26\,000$, and $C(600) = \$10\,000\sqrt{10} \approx \$31\,622.776\,602$. Thus C has an absolute minimum at $x = 150$.

d) Prove that if we define C on the set of all real numbers, then the minimum we found in part a) is still an absolute minimum.

Solution: Finding absolute extrema on domains others than a closed interval is often tricky. In this case, we can prove the statement by looking at C''' more carefully.

$$\begin{aligned} C'''(x) &= \frac{50}{200^2 + x^2} \left(\sqrt{200^2 + x^2} - \frac{x^2}{\sqrt{200^2 + x^2}} \right) = \frac{50}{200^2 + x^2} \left(\frac{(\sqrt{200^2 + x^2})^2}{\sqrt{200^2 + x^2}} - \frac{x^2}{\sqrt{200^2 + x^2}} \right) \\ &= \frac{50}{200^2 + x^2} \left(\frac{200^2 + x^2 - x^2}{\sqrt{200^2 + x^2}} \right) = \frac{50}{200^2 + x^2} \left(\frac{200^2}{\sqrt{200^2 + x^2}} \right) \end{aligned}$$

Since C''' is positive for all x , C' is increasing for all x . So the zero at $x = 150$ is the only zero of C' ; C' is negative on $(-\infty, 150)$ and positive on $(150, \infty)$. Consequently, C is decreasing on $(-\infty, 150)$ and increasing on $(150, \infty)$, and so C has an absolute minimum at $x = 150$.

20. The location function of an object is $s(t) = \frac{120}{1 + 2e^{-t}}$ where t is time, measured in seconds. Where is the object when it is moving with the greatest speed? When is that greatest speed achieved?

Solution:

$$\begin{aligned} s(t) &= \frac{120}{1 + 2e^{-t}} = 120(1 + 2e^{-t})^{-1} \\ v(t) &= s'(t) = 120(-1)(1 + 2e^{-t})^{-2}(2e^{-t})(-1) = \frac{240e^{-t}}{(1 + 2e^{-t})^2} = 240 \frac{e^{-t}}{(1 + 2e^{-t})^2} \end{aligned}$$

We need to find the maximum of v which means we need to differentiate again. This time we will need to use the quotient rule.

$$\begin{aligned} a(t) &= v'(t) = 240 \frac{e^{-t}(-1)(1 + 2e^{-t})^2 - e^{-t}(2)(1 + 2e^{-t})(2e^{-t})(-1)}{(1 + 2e^{-t})^4} = \text{simplify by } (1 + 2e^{-t}) \\ &= 240 \frac{e^{-t}(-1)(1 + 2e^{-t}) - e^{-t}(2)(2e^{-t})(-1)}{(1 + 2e^{-t})^3} \quad \text{factor out } e^{-t} \\ &= 240e^{-t} \frac{-(1 + 2e^{-t}) + 4e^{-t}}{(1 + 2e^{-t})^3} = 240e^{-t} \frac{-1 - 2e^{-t} + 4e^{-t}}{(1 + 2e^{-t})^3} = 240e^{-t} \frac{2e^{-t} - 1}{(1 + 2e^{-t})^3} \end{aligned}$$

$240e^{-t}$ and the denominator are always positive. The only way $v'(t) = 0$ is when $2e^{-t} - 1 = 0$. We solve this equation for t .

$$\begin{aligned} 2e^{-t} - 1 &= 0 & -t &= \ln\left(\frac{1}{2}\right) \\ 2e^{-t} &= 1 & t &= -\ln\left(\frac{1}{2}\right) = \ln 2 \\ e^{-t} &= \frac{1}{2} \end{aligned}$$

Thus $t = \ln 2$ is when the object is fastest. The location of the object is then

$$s(\ln 2) = \frac{120}{1 + 2e^{-\ln 2}} = \frac{120}{1 + 2\left(\frac{1}{2}\right)} = \frac{120}{2} = 60$$

21. Consider the function $g(x) = \frac{-2x}{(x^2 + 1)^2}$.

a) Find all relative extrema of g .

Solution: We differentiate g using the quotient rule.

$$\begin{aligned} g'(x) &= \frac{-2(x^2 + 1)^2 - (-2x)2(x^2 + 1)(2x)}{(x^2 + 1)^4} && \text{simplify by } x^2 + 1 \\ &= \frac{-2(x^2 + 1) - (-2x)2(2x)}{(x^2 + 1)^3} = \frac{-2x^2 - 2 + 8x^2}{(x^2 + 1)^3} = \frac{6x^2 - 2}{(x^2 + 1)^3} \end{aligned}$$

We now factor the numerator to see when g' is positive and negative.

$$g'(x) = \frac{6x^2 - 2}{(x^2 + 1)^3} = \frac{6\left(x^2 - \frac{1}{3}\right)}{(x^2 + 1)^3} = \frac{6\left(x + \frac{1}{\sqrt{3}}\right)\left(x - \frac{1}{\sqrt{3}}\right)}{(x^2 + 1)^3}$$

The denominator is always positive, the numerator is a quadratic expression with a positive leading coefficient, and so g' is positive on $\left(-\infty, -\frac{1}{\sqrt{3}}\right) \cup \left(\frac{1}{\sqrt{3}}, \infty\right)$ and negative on $\left(-\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right)$. Consequently, g is increasing on $\left(-\infty, -\frac{1}{\sqrt{3}}\right)$, decreasing on $\left(-\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right)$, and increasing on $\left(\frac{1}{\sqrt{3}}, \infty\right)$. That means that g has a relative maximum at $x = -\frac{1}{\sqrt{3}}$ and a relative minimum at $x = \frac{1}{\sqrt{3}}$. We compute the function values at $-\frac{1}{\sqrt{3}}$ and $\frac{1}{\sqrt{3}}$.

$$g\left(-\frac{1}{\sqrt{3}}\right) = \frac{-2\left(-\frac{1}{\sqrt{3}}\right)}{\left(\left(-\frac{1}{\sqrt{3}}\right)^2 + 1\right)^2} = \frac{\frac{2}{\sqrt{3}}}{\left(\frac{1}{3} + 1\right)^2} = \frac{\frac{2}{\sqrt{3}}}{\left(\frac{4}{3}\right)^2} = \frac{2}{\sqrt{3}} \cdot \frac{9}{16} = \frac{3\sqrt{3}}{8}$$

This is a good time to notice that g is an odd function

$$g(-x) = \frac{-2(-x)}{\left((-x)^2 + 1\right)^2} = \frac{2x}{(x^2 + 1)^2} = -g(x)$$

and so g has a relative maximum: $\left(-\frac{1}{\sqrt{3}}, \frac{3\sqrt{3}}{8}\right)$ and a relative minimum $\left(-\frac{1}{\sqrt{3}}, \frac{3\sqrt{3}}{8}\right)$.

b) Find all absolute extrema of g .

Solution: We will show that the relative extrema we found in part a) are in fact absolute extrema. This is not a fact that simply follows from the signs of the derivative. For all we know, a decreasing, then increasing, then decreasing function may look like an upside down cubic function that has neither absolute minimum nor absolute maximum.

Claim: $\left(-\frac{1}{\sqrt{3}}, \frac{3\sqrt{3}}{8}\right)$ is an absolute maximum, i.e. for all real numbers x , $g(x) \leq \frac{3\sqrt{3}}{8}$.

In part a) we determined that g is increasing on $\left(-\infty, -\frac{1}{\sqrt{3}}\right)$, decreasing on $\left(-\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right)$, and increasing on $\left(\frac{1}{\sqrt{3}}, \infty\right)$.

Because g is increasing on $\left(-\infty, -\frac{1}{\sqrt{3}}\right)$,

$$\text{for all } x \leq -\frac{1}{\sqrt{3}}, \quad g(x) \leq g\left(-\frac{1}{\sqrt{3}}\right)$$

Because g is decreasing on $\left(-\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right)$, for all $x \leq -\frac{1}{\sqrt{3}} \leq x \leq \frac{1}{\sqrt{3}}$, $g(x) \leq g\left(-\frac{1}{\sqrt{3}}\right)$

We only have to prove that for all $x \geq \frac{1}{\sqrt{3}}$, $g(x) \leq g\left(-\frac{1}{\sqrt{3}}\right)$ is also true, but the increasing/decreasing behavior does not help here, since g is increasing on $\left(\frac{1}{\sqrt{3}}, \infty\right)$. Luckily, there is a simple, elementary way to finish the proof. Recall that

$g(x) = \frac{-2x}{(x^2 + 1)^2}$. It is easy to see that if x is positive, then $g(x)$ is negative. Thus g is negative on $\left(\frac{1}{\sqrt{3}}, \infty\right)$ and so

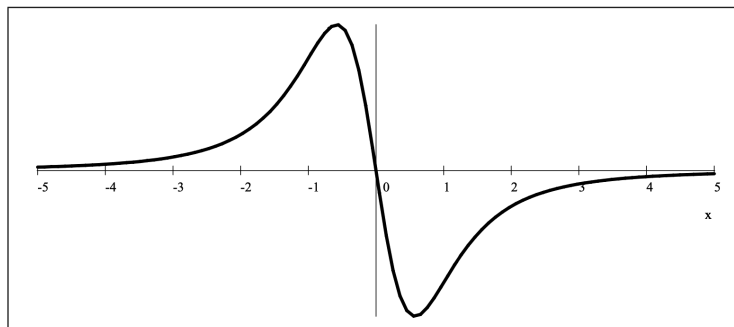
for all $x \geq \frac{1}{\sqrt{3}}$, $g(x) \leq g\left(-\frac{1}{\sqrt{3}}\right) = \frac{3\sqrt{3}}{8}$. Thus g has an absolute maximum at $x = -\frac{1}{\sqrt{3}}$.

Claim: $\left(\frac{1}{\sqrt{3}}, -\frac{3\sqrt{3}}{8}\right)$ is an absolute minimum, i.e. for all real numbers x , $g(x) \geq -\frac{3\sqrt{3}}{8}$.

proof: a very similar argument could work. g is decreasing on $\left(-\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right)$, and increasing on $\left(\frac{1}{\sqrt{3}}, \infty\right)$ and positive on $\left(-\infty, -\frac{1}{\sqrt{3}}\right)$. Another way to prove this is to use the fact that g is an odd function.

$$\begin{aligned} \text{for all } x, g(x) &\leq \frac{3\sqrt{3}}{8} && \text{multiply by } -1 \\ \text{for all } x, -g(x) &\geq -\frac{3\sqrt{3}}{8} && \text{since } g \text{ is odd, } -g(x) = g(-x) \\ \text{for all } x, g(-x) &\geq -\frac{3\sqrt{3}}{8} \end{aligned}$$

and so g has an absolute minimum at $x = \frac{1}{\sqrt{3}}$.



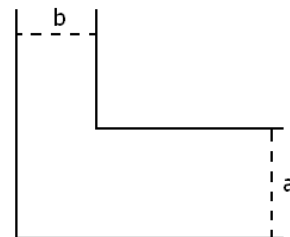
c) Find all values of c for which the function $f(x) = \frac{1}{x^2 + 1} + cx$ is increasing on its entire domain.

Solution: $f(x)$ is always increasing if $f'(x)$ is always positive. $f'(x) = \frac{-2x}{(x^2 + 1)^2} + c$. We proved that the absolute minimum of $\frac{-2x}{(x^2 + 1)^2}$ is $-\frac{3\sqrt{3}}{8}$ and so if we set $c \geq \frac{3\sqrt{3}}{8}$, then $f'(x) = \frac{-2x}{(x^2 + 1)^2} + c$ will be non-negative for all x .

22. Consider the problem of finding the length of the longest rod we can carry through a corner shown on the picture below.

a) For what values of y does the line segment PQ have the shortest length? (Hint: minimize the square of the length.)

Solution: Let x denote the horizontal side of the right triangle with hypotenuse PQ . Using similar triangles,

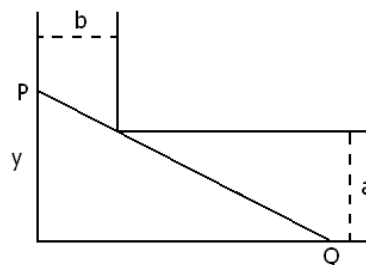


$$\frac{y-a}{b} = \frac{y}{x} \implies x = \frac{by}{y-a}$$

The square of the length is (by the Pythagorean Theorem) is on domain $(0, \infty)$

$$f(y) = x^2 + y^2 = \left(\frac{by}{y-a}\right)^2 + y^2 = y^2 + b^2 \left(\frac{y}{y-a}\right)^2$$

$$f'(y) = \left(y^2 + b^2 \frac{y^2}{(y-a)^2}\right)' = 2y - \frac{2ab^2y}{(y-a)^3}$$



$$f'(y) = 0$$

$$2y - \frac{2ab^2y}{(y-a)^3} = 0$$

$$2y \left(1 - \frac{ab^2}{(y-a)^3}\right) = 0 \implies y = 0 \text{ or } 1 - \frac{ab^2}{(y-a)^3} = 0$$

$$1 - \frac{ab^2}{(y-a)^3} = 0$$

$$\sqrt[3]{ab^2} = y - a$$

$$\frac{ab^2}{(y-a)^3} = 1$$

$$y = \sqrt[3]{ab^2} + a$$

$$ab^2 = (y-a)^3$$

We need to make sure that $y = \sqrt[3]{ab^2} + a$ is indeed an absolute minimum.

b) What is the length of the longest ladder which can be slid along the floor from a corridor of width a to a corridor of width b ?

Solution: We simply need to evaluate $L(y) = \sqrt{f(y)}$ at $y = \sqrt[3]{ab^2} + a = a^{1/3}b^{2/3} + a$

$$\begin{aligned}
 \sqrt{f(y)} &= \sqrt{y^2 + b^2 \left(\frac{y}{y-a} \right)^2} = y \sqrt{1 + \frac{b^2}{(y-a)^2}} \\
 \sqrt{f(a^{1/3}b^{2/3} + a)} &= (a^{1/3}b^{2/3} + a) \sqrt{1 + \frac{b^2}{(a^{1/3}b^{2/3} + a - a)^2}} = (a^{1/3}b^{2/3} + a) \sqrt{1 + \frac{b^2}{(a^{1/3}b^{2/3})^2}} \\
 &= a^{1/3} (b^{2/3} + a^{2/3}) \sqrt{1 + \frac{b^2}{a^{2/3}b^{4/3}}} = (b^{2/3} + a^{2/3}) \sqrt{a^{2/3} \left(1 + \frac{b^{2-4/3}}{a^{2/3}} \right)} \\
 &= (b^{2/3} + a^{2/3}) \sqrt{a^{2/3} \left(1 + \frac{b^{2/3}}{a^{2/3}} \right)} = (b^{2/3} + a^{2/3}) \sqrt{a^{2/3} + b^{2/3}} \\
 &= (a^{2/3} + b^{2/3})^{3/2} = \sqrt{\left(\sqrt[3]{a^2} + \sqrt[3]{b^2} \right)^3}
 \end{aligned}$$