

Sample Problems

Compute each of the following integrals.

1. $\int \frac{1}{x^2 - 4} dx$

5. $\int \frac{x^2 + x - 3}{(x + 1)(x - 2)(x - 5)} dx$

9. $\int \sec x dx$

2. $\int \frac{2x}{(x + 3)(3x + 1)} dx$

6. $\int \frac{2x - 1}{(x - 5)^2} dx$

10. $\int \operatorname{csch} x dx$

3. $\int \frac{x + 5}{x^2 - 2x - 3} dx$

7. $\int \frac{x + 3}{(x - 1)^3} dx$

4. $\int \frac{x^4 + x^3 - 5x^2 + 26x - 21}{x^2 + 3x - 4} dx$

8. $\int \frac{x^4}{x^4 - 1} dx$

Practice Problems

1. $\int \frac{1}{x^2 + 3x} dx$

7. $\int \frac{2x^3 - x^2 - 10x - 4}{x^2 - 4} dx$

12. $\int \frac{2x + 1}{x^2 + 1} dx$

2. $\int \frac{x - 5}{x^2 - 2x - 8} dx$

8. $\int \frac{5x - 17}{x^2 - 6x + 9} dx$

13. $\int \frac{x^2 + 2}{x(x^2 + 6)} dx$

3. $\int \frac{1}{x^2 - a^2} dx$

9. $\int \frac{2x^2 + 7x + 3}{x^2 + 1} dx$

14. $\int \frac{-x + 6}{(x + 3)^2} dx$

4. $\int \frac{x - 1}{x^2 - 4} dx$

10. $\int \frac{2x^2 - x + 20}{(x - 2)(x^2 + 9)} dx$

15. $\int \frac{2x - 3}{x^2 + 9} dx$

5. $\int \frac{x - 1}{x^2 + 4} dx$

11. $\int \frac{x^4}{x^4 - 16} dx$

16. $\int \frac{x^2 + 2x - 1}{x^3 - x} dx$

Sample Problems - Answers

- 1.) $\frac{1}{4} \ln|x-2| - \frac{1}{4} \ln|x+2| + C$ 2.) $\frac{3}{4} \ln|x+3| - \frac{1}{12} \ln|3x+1| + C$ 3.) $2 \ln|x-3| - \ln|x+1| + C$
- 4.) $\frac{x^3}{3} - x^2 + 5x + \frac{13}{5} \ln|x+4| + \frac{2}{5} \ln|x-1| + C$ 5.) $-\frac{1}{6} \ln|x+1| - \frac{1}{3} \ln|x-2| + \frac{2}{3} \ln|x-5| + C$
- 6.) $2 \ln|x-5| - \frac{9}{x-5} + C$ 7.) $-\frac{1}{x-1} - \frac{2}{(x-1)^2} + C = \frac{-x-1}{(x-1)^2} + C$
- 8.) $x - \frac{1}{2} \arctan x - \frac{1}{4} \ln|x+1| + \frac{1}{4} \ln|x-1| + C$ 9.) $\ln|\sec x + \tan x| + C = -\ln|\sec x - \tan x| + C$
- 10.) $\ln|e^x - 1| - \ln(e^x + 1) + C$

Practice Problems - Answers

- 1.) $\frac{1}{3} \ln|x| - \frac{1}{3} \ln|x+3| + C$ 2.) $\frac{7}{6} \ln|x+2| - \frac{1}{6} \ln|x-4| + C$ 3.) $\frac{1}{2a} \ln|x-a| - \frac{1}{2a} \ln|x+a| + C$
- 4.) $\frac{1}{4} \ln|x-2| + \frac{3}{4} \ln|x+2| + C$ 5.) $\frac{1}{2} \ln(x^2+4) - \frac{1}{2} \tan^{-1} \frac{1}{2}x + C$
- 6.) $x + \frac{1}{4} \ln|x-1| - \frac{9}{4} \ln|x+3| + C$ 7.) $x^2 - x - 3 \ln|x-2| + \ln|x+2| + C$
- 8.) $5 \ln|x-3| + \frac{2}{x-3} + C$ 9.) $2x + \frac{7}{2} \ln(x^2+1) + \tan^{-1} x + C$
- 10.) $2 \ln|x-2| - \frac{1}{3} \tan^{-1} \frac{x}{3} + C$ 11.) $x + \frac{1}{2} \ln|x-2| - \frac{1}{2} \ln|x+2| - \tan^{-1} \frac{x}{2} + C$
- 12.) $\tan^{-1} x + \ln(x^2+1) + C$ 13.) $\frac{1}{3} \ln|x^3+6x| + C$ 14.) $-\ln|x+3| - \frac{9}{x+3} + C$
- 15.) $\ln(x^2+9) - \tan^{-1} \frac{x}{3} + C$ 16.) $\ln|x| + \ln|x-1| - \ln|x+1| + C$

Sample Problems - Solutions

Compute each of the following integrals.

1. $\int \frac{1}{x^2 - 4} dx$

Solution: We factor the denominator: $x^2 - 4 = (x + 2)(x - 2)$. Next, we re-write the fraction $\frac{1}{x^2 - 4}$ as a sum (or difference) of fractions with denominators $x + 2$ and $x - 2$. This means that we need to solve for A and B in the equation

$$\frac{A}{x + 2} + \frac{B}{x - 2} = \frac{1}{x^2 - 4}$$

To simplify the left-hand side, we bring the fractions to the common denominator:

$$\frac{A(x - 2)}{(x + 2)(x - 2)} + \frac{B(x + 2)}{(x - 2)(x + 2)} = \frac{Ax - 2A + Bx + 2B}{x^2 - 4} = \frac{(A + B)x - 2A + 2B}{x^2 - 4}$$

Thus we have

$$\frac{(A + B)x - 2A + 2B}{x^2 - 4} = \frac{1}{x^2 - 4}$$

We clear the denominators by multiplication

$$(A + B)x - 2A + 2B = 1$$

The equation above is about two polynomials: they are equal to each other as functions and so they must be identical, coefficient by coefficient. In other words,

$$(A + B)x - 2A + 2B = 0x + 1$$

This gives us an equation for each coefficient, forming a system of linear equations:

$$\begin{aligned} A + B &= 0 \\ -2A + 2B &= 1 \end{aligned}$$

We solve this system and obtain $A = -\frac{1}{4}$ and $B = \frac{1}{4}$.

So our fraction, $\frac{1}{x^2 - 4}$ can be re-written as $\frac{-\frac{1}{4}}{x + 2} + \frac{\frac{1}{4}}{x - 2}$. We check:

$$\begin{aligned} \frac{-\frac{1}{4}}{x + 2} + \frac{\frac{1}{4}}{x - 2} &= \frac{-\frac{1}{4}(x - 2)}{(x + 2)(x - 2)} + \frac{\frac{1}{4}(x + 2)}{(x - 2)(x + 2)} = \frac{-\frac{1}{4}(x - 2) + \frac{1}{4}(x + 2)}{(x + 2)(x - 2)} = \frac{-\frac{1}{4}x + \frac{1}{2} + \frac{1}{4}x + \frac{1}{2}}{(x + 2)(x - 2)} \\ &= \frac{1}{x^2 - 4} \end{aligned}$$

Now we can easily integrate:

$$\int \frac{1}{x^2 - 4} dx = \int \frac{-\frac{1}{4}}{x + 2} + \frac{\frac{1}{4}}{x - 2} dx = -\frac{1}{4} \int \frac{1}{x + 2} dx + \frac{1}{4} \int \frac{1}{x - 2} dx = \boxed{-\frac{1}{4} \ln |x + 2| + \frac{1}{4} \ln |x - 2| + C}$$

Method 2: The values of A and B can be found using a slightly different method as follows. Consider first the equation

$$\frac{A}{x+2} + \frac{B}{x-2} = \frac{1}{x^2-4}$$

We bring the fractions to the common denominator:

$$\frac{A(x-2)}{(x+2)(x-2)} + \frac{B(x+2)}{(x-2)(x+2)} = \frac{1}{x^2-4}$$

and then multiply both sides by the denominator:

$$A(x-2) + B(x+2) = 1$$

The equation above is about two functions; the two sides must be equal for all values of x . Let us substitute $x = 2$ into both sides:

$$\begin{aligned} A(0) + B(4) &= 1 \\ B &= \frac{1}{4} \end{aligned}$$

Let us substitute $x = -2$ into both sides:

$$\begin{aligned} A(-4) + B(0) &= 1 \\ A &= -\frac{1}{4} \end{aligned}$$

and so $A = -\frac{1}{4}$ and $B = \frac{1}{4}$.

$$2. \int \frac{2x}{(x+3)(3x+1)} dx$$

Solution: We re-write the fraction $\frac{2x}{(x+3)(3x+1)}$ as a sum (or difference) of fractions with denominators $x+3$ and $3x+1$. This means that we need to solve for A and B in the equation

$$\frac{A}{x+3} + \frac{B}{3x+1} = \frac{2x}{(x+3)(3x+1)}$$

To simplify the left-hand side, we bring the fractions to the common denominator:

$$\begin{aligned} \frac{A}{x+3} + \frac{B}{3x+1} &= \frac{A(3x+1)}{(x+3)(3x+1)} + \frac{B(x+3)}{(x+3)(3x+1)} = \frac{A(3x+1) + B(x+3)}{(x+3)(3x+1)} = \frac{3Ax + A + Bx + 3B}{(x+3)(3x+1)} \\ &= \frac{(3A+B)x + A + 3B}{(x+3)(3x+1)} \end{aligned}$$

Thus we have

$$\frac{(3A+B)x + A + 3B}{(x+3)(3x+1)} = \frac{2x}{(x+3)(3x+1)}$$

We clear the denominators by multiplication

$$(3A+B)x + A + 3B = 2x$$

The equation above is about two polynomials: they are equal to each other as functions and so they must be identical, coefficient by coefficient. In other words,

$$(3A + B)x + A + 3B = 2x + 0$$

This gives us an equation for each coefficient, forming a system of linear equations:

$$\begin{aligned} 3A + B &= 2 \\ A + 3B &= 0 \end{aligned}$$

We solve this system and obtain $A = \frac{3}{4}$ and $B = -\frac{1}{4}$.

So our fraction, $\frac{2x}{(x+3)(3x+1)}$ can be re-written as $\frac{\frac{3}{4}}{x+3} + \frac{-\frac{1}{4}}{3x+1}$. We check:

$$\begin{aligned} \frac{\frac{3}{4}}{x+3} + \frac{-\frac{1}{4}}{3x+1} &= \frac{\frac{3}{4}(3x+1)}{(x+3)(3x+1)} + \frac{-\frac{1}{4}(x+3)}{(x+3)(3x+1)} = \frac{\frac{3}{4}(3x+1) - \frac{1}{4}(x+3)}{(x+3)(3x+1)} = \frac{\frac{9}{4}x + \frac{3}{4} - \frac{1}{4}x - \frac{3}{4}}{(x+3)(3x+1)} \\ &= \frac{2x}{(x+3)(3x+1)} \end{aligned}$$

Now we can easily integrate:

$$\int \frac{2x}{(x+3)(3x+1)} dx = \int \frac{\frac{3}{4}}{x+3} + \frac{-\frac{1}{4}}{3x+1} dx = \frac{3}{4} \int \frac{1}{x+3} dx - \frac{1}{4} \int \frac{1}{3x+1} dx = \boxed{\frac{3}{4} \ln|x+3| - \frac{1}{12} \ln|3x+1| + C}$$

The second integral can be computed using the substitution $u = 3x + 1$.

Method 2: The values of A and B can be found using a slightly different method as follows. Consider first the equation

$$\frac{A}{x+3} + \frac{B}{3x+1} = \frac{2x}{(x+3)(3x+1)}$$

We bring the fractions to the common denominator:

$$\frac{A(3x+1)}{(x+3)(3x+1)} + \frac{B(x+3)}{(x+3)(3x+1)} = \frac{2x}{(x+3)(3x+1)}$$

and then multiply both sides by the denominator:

$$A(3x+1) + B(x+3) = 2x$$

The equation above is about two functions; the two sides must be equal for all values of x . Let us substitute $x = -\frac{1}{3}$ into both sides:

$$\begin{aligned} A(0) + B\left(-\frac{1}{3} + 3\right) &= 2\left(-\frac{1}{3}\right) \\ \frac{8}{3}B &= -\frac{2}{3} \\ B &= -\frac{1}{4} \end{aligned}$$

Let us substitute $x = -3$ into both sides:

$$\begin{aligned} A(3(-3) + 1) + B(-3 + 3) &= 2(-3) \\ -8A + B(0) &= -6 \\ -8A &= -6 \\ A &= \frac{3}{4} \end{aligned}$$

and so $A = \frac{3}{4}$ and $B = -\frac{1}{4}$.

$$3. \int \frac{x+5}{x^2-2x-3} dx$$

Solution: We factor the denominator: $x^2-2x-3 = (x+1)(x-3)$. Next, we re-write the fraction $\frac{x+5}{x^2-2x-3}$ as a sum (or difference) of fractions with denominators $x+1$ and $x-3$. This means that we need to solve for A and B in the equation

$$\frac{A}{x+1} + \frac{B}{x-3} = \frac{x+5}{x^2-2x-3}$$

To simplify the left-hand side, we bring the fractions to the common denominator:

$$\frac{A(x-3)}{(x+1)(x-3)} + \frac{B(x+1)}{(x-3)(x+1)} = \frac{Ax-3A+Bx+B}{x^2-2x-3} = \frac{(A+B)x-3A+B}{x^2-2x-3}$$

Thus

$$\frac{(A+B)x-3A+B}{x^2-2x-3} = \frac{x+5}{x^2-2x-3}$$

We clear the denominators by multiplication

$$(A+B)x-3A+B = x+5$$

The equation above is about two polynomials: they are equal to each other as functions and so they must be identical, coefficient by coefficient. This gives us an equation for each coefficient, forming a system of linear equations:

$$\begin{aligned} A+B &= 1 \\ -3A+B &= 5 \end{aligned}$$

We solve the system and obtain $A = -1$ and $B = 2$.

So we have that our fraction, $\frac{x+5}{x^2-2x-3}$ can be re-written as $\frac{-1}{x+1} + \frac{2}{x-3}$. We check:

$$\frac{-1}{x+1} + \frac{2}{x-3} = \frac{-1(x-3)}{(x+1)(x-3)} + \frac{2(x+1)}{(x-3)(x+1)} = \frac{-(x-3)+2(x+1)}{(x+1)(x-3)} = \frac{-x+3+2x+2}{x^2-2x-3} = \frac{x+5}{x^2-2x-3}$$

Now we can easily integrate:

$$\int \frac{x+5}{x^2-2x-3} dx = \int \frac{-1}{x+1} + \frac{2}{x-3} dx = -\int \frac{1}{x+1} dx + 2 \int \frac{1}{x-3} dx = \boxed{-\ln|x+1| + 2\ln|x-3| + C}$$

Method 2: The values of A and B can be found using a slightly different method as follows. Consider first the equation

$$\frac{A}{x+1} + \frac{B}{x-3} = \frac{x+5}{x^2-2x-3}$$

We bring the fractions to the common denominator:

$$\frac{A(x-3)}{(x-3)(x+1)} + \frac{B(x+1)}{(x-3)(x+1)} = \frac{x+5}{x^2-2x-3}$$

and then multiply both sides by the denominator:

$$A(x-3) + B(x+1) = x+5$$

The equation above is about two functions; the two sides must be equal for all values of x . Let us substitute $x = 3$ into both sides:

$$\begin{aligned} A(0) + B(4) &= 3 + 5 \\ 4B &= 8 \\ B &= 2 \end{aligned}$$

Let us substitute $x = -1$ into both sides:

$$\begin{aligned} A(-4) + B(0) &= -1 + 5 \\ -4A &= 4 \\ A &= -1 \end{aligned}$$

and so $A = -1$ and $B = 2$.

$$4. \int \frac{x^4 + x^3 - 5x^2 + 26x - 21}{x^2 + 3x - 4} dx$$

Solution: This rational function is an improper fraction since the numerator has a higher degree than the denominator. We first perform long division. This process is similar to long division among numbers. For example, to simplify $\frac{38}{7}$, we perform the long division $38 \div 7 = 5 \text{ R } 3$ which is the same thing as to say that $\frac{38}{7} = 5\frac{3}{7}$. The division:

$$\begin{array}{r} x^2 + 3x - 4 \overline{) \begin{array}{r} x^4 + x^3 - 5x^2 + 26x - 21 \\ -x^4 - 3x^3 + 4x^2 \\ \hline -2x^3 - x^2 + 26x - 21 \\ 2x^3 + 6x^2 - 8x \\ \hline 5x^2 + 18x - 21 \\ -5x^2 - 15x + 20 \\ \hline 3x - 1 \end{array}} \end{array}$$

$$\text{Step 1: } \frac{x^4}{x^2} = x^2$$

$$x^2(x^2 + 3x - 4) = x^4 + 3x^3 - 4x^2$$

$$-(x^4 + 3x^3 - 4x^2) = -x^4 - 3x^3 + 4x^2$$

We add that to the original polynomial shown above.

$$\text{Step 2: } \frac{-2x^3}{x^2} = -2x$$

$$-2x(x^2 + 3x - 4) = -2x^3 - 6x^2 + 8x$$

$$-1(-2x^3 - 6x^2 + 8x) = 2x^3 + 6x^2 - 8x$$

We add that to the original polynomial shown above.

$$\text{Step 3: } \frac{5x^2}{x^2} = 5$$

$$5(x^2 + 3x - 4) = 5x^2 + 15x - 20$$

$$-1(5x^2 + 15x - 20) = -5x^2 - 15x + 20$$

We add that to the original polynomial shown above.

The result of this computation is that

$$\frac{x^4 + x^3 - 5x^2 + 26x - 21}{x^2 + 3x - 4} = x^2 - 2x + 5 + \frac{3x - 1}{x^2 + 3x - 4}$$

very much like $\frac{38}{7} = 5 + \frac{3}{7}$. Thus

$$\begin{aligned} \int \frac{x^4 + x^3 - 5x^2 + 26x - 21}{x^2 + 3x - 4} dx &= \int x^2 - 2x + 5 + \frac{3x - 1}{x^2 + 3x - 4} dx = \int x^2 - 2x + 5 dx + \int \frac{3x - 1}{x^2 + 3x - 4} dx \\ &= \frac{x^3}{3} - x^2 + 5x + C_1 + \int \frac{3x - 1}{x^2 + 3x - 4} dx \end{aligned}$$

We apply the method of partial fractions to compute $\int \frac{3x - 1}{x^2 + 3x - 4} dx$.

We factor the denominator: $x^2 + 3x - 4 = (x + 4)(x - 1)$. Next, we re-write the fraction $\frac{3x - 1}{x^2 + 3x - 4}$ as a sum (or difference) of fractions with denominators $x + 4$ and $x - 1$. This means that we need to solve for A and B in the equation

$$\frac{A}{x + 4} + \frac{B}{x - 1} = \frac{3x - 1}{x^2 + 3x - 4}$$

To simplify the left-hand side, we bring the fractions to the common denominator:

$$\frac{A(x - 1)}{(x + 4)(x - 1)} + \frac{B(x + 4)}{(x + 4)(x - 1)} = \frac{Ax - A + Bx + 4B}{x^2 + 3x - 4} = \frac{(A + B)x - A + 4B}{x^2 + 3x - 4}$$

Thus

$$\frac{(A + B)x - A + 4B}{x^2 + 3x - 4} = \frac{3x - 1}{x^2 + 3x - 4}$$

We clear the denominators by multiplication

$$(A + B)x - A + 4B = 3x - 1$$

The equation above is about two polynomials: they are equal to each other as functions and so they must be identical, coefficient by coefficient. This gives us an equation for each coefficient that forms a system of linear equations:

$$\begin{aligned} A + B &= 3 \\ -A + 4B &= -1 \end{aligned}$$

We solve the system and obtain $A = \frac{13}{5}$ and $B = \frac{2}{5}$.

So our fraction, $\frac{3x - 1}{x^2 + 3x - 4}$ can be re-written as $\frac{\frac{13}{5}}{x + 4} + \frac{\frac{2}{5}}{x - 1}$. We check:

$$\begin{aligned} \frac{\frac{13}{5}}{x + 4} + \frac{\frac{2}{5}}{x - 1} &= \frac{\frac{13}{5}(x - 1)}{(x + 4)(x - 1)} + \frac{\frac{2}{5}(x + 4)}{(x - 1)(x + 4)} = \frac{\frac{13}{5}(x - 1) + \frac{2}{5}(x + 4)}{(x + 4)(x - 1)} \\ &= \frac{\frac{13}{5}x - \frac{13}{5} + \frac{2}{5}x + \frac{8}{5}}{(x + 4)(x - 1)} = \frac{\frac{15}{5}x - \frac{5}{5}}{x^2 + 3x - 4} = \frac{3x - 1}{x^2 + 3x - 4} \end{aligned}$$

Now we can easily integrate:

$$\begin{aligned}\int \frac{3x-1}{x^2+3x-4} dx &= \int \frac{\frac{13}{5}}{x+4} + \frac{\frac{2}{5}}{x-1} dx = \int \frac{\frac{13}{5}}{x+4} dx + \int \frac{\frac{2}{5}}{x-1} dx \\ &= \frac{13}{5} \int \frac{1}{x+4} dx + \frac{2}{5} \int \frac{1}{x-1} dx = \frac{13}{5} \ln|x+4| + \frac{2}{5} \ln|x-1| + C\end{aligned}$$

Thus the final answer is

$$\begin{aligned}\int \frac{x^4 + x^3 - 5x^2 + 26x - 21}{x^2 + 3x - 4} dx &= \\ &= \frac{x^3}{3} - x^2 + 5x + C_1 + \int \frac{3x-1}{x^2+3x-4} dx = \frac{x^3}{3} - x^2 + 5x + C_1 + \frac{13}{5} \ln|x+4| + \frac{2}{5} \ln|x-1| + C_2 \\ &= \boxed{\frac{x^3}{3} - x^2 + 5x + \frac{13}{5} \ln|x+4| + \frac{2}{5} \ln|x-1| + C}\end{aligned}$$

Method 2: The values of A and B can be found using a slightly different method as follows. Consider first the equation

$$\frac{A}{x+4} + \frac{B}{x-1} = \frac{3x-1}{x^2+3x-4}$$

We bring the fractions to the common denominator:

$$\frac{A(x-1)}{(x+4)(x-1)} + \frac{B(x+4)}{(x+4)(x-1)} = \frac{3x-1}{(x+4)(x-1)}$$

and then multiply both sides by the denominator:

$$A(x-1) + B(x+4) = 3x-1$$

The equation above is about two functions; the two sides must be equal for all values of x . Let us substitute $x = 1$ into both sides:

$$\begin{aligned}A(1-1) + B(1+4) &= 3 \cdot 1 - 1 \\ A \cdot 0 + B \cdot 5 &= 3 \cdot 1 - 1 \\ 5B &= 2 \\ B &= \frac{2}{5}\end{aligned}$$

Let us substitute $x = -4$ into both sides:

$$\begin{aligned}A(-4-1) + B(-4+4) &= 3(-4) - 1 \\ -5A &= -13 \\ A &= \frac{13}{5}\end{aligned}$$

and so $A = \frac{13}{5}$ and $B = \frac{2}{5}$.

$$5. \int \frac{x^2 + x - 3}{(x+1)(x-2)(x-5)} dx$$

Solution: We re-write the fraction $\frac{x^2 + x - 3}{(x+1)(x-2)(x-5)}$ as a sum (or difference) of fractions with denominators $x+1$, $x-2$ and $x-5$. This means that we need to solve for A , B , and C in the equation

$$\frac{A}{x+1} + \frac{B}{x-2} + \frac{C}{x-5} = \frac{x^2 + x - 3}{(x+1)(x-2)(x-5)}$$

To simplify the left-hand side, we bring the fractions to the common denominator:

$$\begin{aligned} \frac{A}{x+1} + \frac{B}{x-2} + \frac{C}{x-5} &= \frac{A(x-2)(x-5)}{(x+1)(x-2)(x-5)} + \frac{B(x+1)(x-5)}{(x+1)(x-2)(x-5)} + \frac{C(x+1)(x-2)}{(x+1)(x-2)(x-5)} \\ &= \frac{A(x^2 - 7x + 10)}{(x+1)(x-2)(x-5)} + \frac{B(x^2 - 4x - 5)}{(x+1)(x-2)(x-5)} + \frac{C(x^2 - x - 2)}{(x+1)(x-2)(x-5)} \\ &= \frac{A(x^2 - 7x + 10) + B(x^2 - 4x - 5) + C(x^2 - x - 2)}{(x+1)(x-2)(x-5)} \\ &= \frac{Ax^2 - 7Ax + 10A + Bx^2 - 4Bx - 5B + Cx^2 - Cx - 2C}{(x+1)(x-2)(x-5)} \\ &= \frac{(A+B+C)x^2 + (-7A-4B-C)x + 10A-5B-2C}{(x+1)(x-2)(x-5)} \end{aligned}$$

Thus

$$\frac{(A+B+C)x^2 + (-7A-4B-C)x + 10A-5B-2C}{(x+1)(x-2)(x-5)} = \frac{x^2 + x - 3}{(x+1)(x-2)(x-5)}$$

We clear the denominators by multiplication

$$(A+B+C)x^2 + (-7A-4B-C)x + 10A-5B-2C = x^2 + x - 3$$

The equation above is about two polynomials: they are equal to each other as functions and so they must be identical, coefficient by coefficient. We have an equation for each coefficient that gives us a system of linear equations:

$$\begin{array}{rrrrrcl} A & + & B & + & C & = & 1 \\ -7A & - & 4B & - & C & = & 1 \\ 10A & - & 5B & - & 2C & = & -3 \end{array}$$

We solve the system by elimination: first we will eliminate C from the second and third equations. To eliminate C from the second equation, we simply add the first and second equations.

$$\begin{array}{rrrrrcl} A & + & B & + & C & = & 1 \\ -7A & - & 4B & - & C & = & 1 \\ \hline -6A & - & 3B & & & = & 2 \end{array}$$

To eliminate C from the third equation, we multiply the first equation by 2 and add that to the third equation.

$$\begin{array}{rrrrrcl} 2A & + & 2B & + & 2C & = & 2 \\ 10A & - & 5B & - & 2C & = & -3 \\ \hline 12A & - & 3B & & & = & -1 \end{array}$$

We now have a system of linear equations in two variables:

$$\begin{aligned}-6A - 3B &= 2 \\ 12A - 3B &= -1\end{aligned}$$

We will eliminate B by adding the opposite of the first equation to the second equation.

$$\begin{array}{r} 6A + 3B = -2 \\ 12A - 3B = -1 \\ \hline 18A = -3 \\ A = -\frac{1}{6} \end{array}$$

Using the equation $6A + 3B = -2$ we can now solve for B .

$$\begin{aligned} 6\left(-\frac{1}{6}\right) + 3B &= -2 \\ -1 + 3B &= -2 \\ 3B &= -1 \\ B &= -\frac{1}{3} \end{aligned}$$

Using the first equation, we can now solve for C .

$$\begin{aligned} A + B + C &= 1 \\ -\frac{1}{6} + \left(-\frac{1}{3}\right) + C &= 1 \\ -\frac{1}{2} + C &= 1 \\ C &= \frac{3}{2} \end{aligned}$$

Thus $A = -\frac{1}{6}$, $B = -\frac{1}{3}$, and $C = \frac{3}{2}$

So we have that our fraction, $\frac{x^2 + x - 3}{(x+1)(x-2)(x-5)}$ can be re-written as $\frac{-\frac{1}{6}}{x+1} + \frac{-\frac{1}{3}}{x-2} + \frac{\frac{3}{2}}{x-5}$. We check:

$$\begin{aligned} \frac{-\frac{1}{6}}{x+1} + \frac{-\frac{1}{3}}{x-2} + \frac{\frac{3}{2}}{x-5} &= \frac{-\frac{1}{6}(x-2)(x-5)}{(x+1)(x-2)(x-5)} + \frac{-\frac{1}{3}(x+1)(x-5)}{(x+1)(x-2)(x-5)} + \frac{\frac{3}{2}(x+1)(x-2)}{(x+1)(x-2)(x-5)} \\ &= \frac{-\frac{1}{6}(x-2)(x-5) - \frac{1}{3}(x+1)(x-5) + \frac{3}{2}(x+1)(x-2)}{(x+1)(x-2)(x-5)} \\ &= \frac{-\frac{1}{6}(x^2 - 7x + 10) - \frac{1}{3}(x^2 - 4x - 5) + \frac{3}{2}(x^2 - x - 2)}{(x+1)(x-2)(x-5)} \\ &= \frac{-\frac{1}{6}x^2 + \frac{7}{6}x - \frac{5}{3} - \frac{1}{3}x^2 + \frac{4}{3}x + \frac{5}{3} + \frac{3}{2}x^2 - \frac{3}{2}x - 3}{(x+1)(x-2)(x-5)} \\ &= \frac{\left(-\frac{1}{6} - \frac{1}{3} + \frac{3}{2}\right)x^2 + \left(\frac{7}{6} + \frac{4}{3} - \frac{3}{2}\right)x - \frac{5}{3} + \frac{5}{3} - 3}{(x+1)(x-2)(x-5)} \\ &= \frac{x^2 + x - 3}{(x+1)(x-2)(x-5)} \end{aligned}$$

Now we can easily integrate:

$$\begin{aligned}
 \int \frac{x^2 + x - 3}{(x+1)(x-2)(x-5)} dx &= \int \frac{-\frac{1}{6}}{x+1} + \frac{-\frac{1}{3}}{x-2} + \frac{\frac{3}{2}}{x-5} dx \\
 &= -\frac{1}{6} \int \frac{1}{x+1} dx - \frac{1}{3} \int \frac{1}{x-2} dx + \frac{3}{2} \int \frac{1}{x-5} dx \\
 &= \boxed{-\frac{1}{6} \ln|x+1| - \frac{1}{3} \ln|x-2| + \frac{2}{3} \ln|x-5| + C}
 \end{aligned}$$

Method 2: The values of A , B , and C can be found using a slightly different method as follows. Consider first the equation

$$\frac{A}{x+1} + \frac{B}{x-2} + \frac{C}{x-5} = \frac{x^2 + x - 3}{(x+1)(x-2)(x-5)}$$

We bring the fractions to the common denominator:

$$\frac{A(x-2)(x-5)}{(x+1)(x-2)(x-5)} + \frac{B(x+1)(x-5)}{(x+1)(x-2)(x-5)} + \frac{C(x+1)(x-2)}{(x+1)(x-2)(x-5)} = \frac{x^2 + x - 3}{(x+1)(x-2)(x-5)}$$

and then multiply both sides by the denominator:

$$A(x-2)(x-5) + B(x+1)(x-5) + C(x+1)(x-2) = x^2 + x - 3$$

The equation above is about two functions; the two sides must be equal for all values of x . Let us substitute $x = 2$ into both sides:

$$\begin{aligned}
 A(2-2)(2-5) + B(2+1)(2-5) + C(2+1)(2-2) &= 2^2 + 2 - 3 \\
 0A - 9B + 0C &= 3 \\
 -9B &= 3 \\
 B &= -\frac{1}{3}
 \end{aligned}$$

Let us substitute $x = -1$ into both sides:

$$\begin{aligned}
 A(-1-2)(-1-5) + B(-1+1)(-1-5) + C(-1+1)(-1-2) &= (-1)^2 + (-1) - 3 \\
 A(-3)(-6) + 0B + 0C &= -3 \\
 18A &= -3 \\
 A &= -\frac{1}{6}
 \end{aligned}$$

Let us substitute $x = 5$ into both sides:

$$\begin{aligned}
 A(5-2)(5-5) + B(5+1)(5-5) + C(5+1)(5-2) &= 5^2 + 5 - 3 \\
 A(0) + B(0) + C(6)(3) &= 27 \\
 18C &= 27 \\
 C &= \frac{3}{2}
 \end{aligned}$$

and so $A = -\frac{1}{6}$, $B = -\frac{1}{3}$, and $C = \frac{2}{3}$.

$$6. \int \frac{2x-1}{(x-5)^2} dx$$

Solution: We will re-write the fraction $\frac{2x-1}{(x-5)^2}$ as a sum (or difference) of fractions with denominators $x-5$ and $(x-5)^2$. This means that we need to solve for A and B in the equation

$$\frac{A}{x-5} + \frac{B}{(x-5)^2} = \frac{2x-1}{(x-5)^2}$$

To simplify the left-hand side, we bring the fractions to the common denominator:

$$\frac{A}{x-5} + \frac{B}{(x-5)^2} = \frac{A(x-5)}{(x-5)^2} + \frac{B}{(x-5)^2} = \frac{A(x-5) + B}{(x-5)^2} = \frac{Ax - 5A + B}{(x-5)^2}$$

Thus we have

$$\frac{Ax - 5A + B}{(x-5)^2} = \frac{2x-1}{(x-5)^2}$$

We clear the denominators by multiplication

$$Ax - 5A + B = 2x - 1$$

The equation above is about two polynomials: they are equal to each other as functions and so they must be identical, coefficient by coefficient. This gives us an equation for each coefficient, forming a system of linear equations:

$$\begin{aligned} A &= 2 \\ -5A + B &= -1 \end{aligned}$$

We solve this system and obtain $A = 2$ and $B = 9$.

So our fraction, $\frac{2x-1}{(x-5)^2}$ can be re-written as $\frac{2}{x-5} + \frac{9}{(x-5)^2}$. We check:

$$\frac{2}{x-5} + \frac{9}{(x-5)^2} = \frac{2(x-5)}{(x-5)^2} + \frac{9}{(x-5)^2} = \frac{2x-10+9}{(x-5)^2} = \frac{2x-1}{(x-5)^2}$$

Now we can easily integrate:

$$\int \frac{2x-1}{(x-5)^2} dx = \int \frac{2}{x-5} + \frac{9}{(x-5)^2} dx = 2 \int \frac{1}{x-5} dx + 9 \int \frac{1}{(x-5)^2} dx = \boxed{2 \ln |x-5| - \frac{9}{x-5} + C}$$

Method 2: The values of A and B can be found using a slightly different method as follows. Consider first the equation

$$\frac{A}{x-5} + \frac{B}{(x-5)^2} = \frac{2x-1}{(x-5)^2}$$

We bring the fractions to the common denominator:

$$\frac{A(x-5)}{(x-5)^2} + \frac{B}{(x-5)^2} = \frac{2x-1}{(x-5)^2}$$

and then multiply both sides by the denominator:

$$A(x-5) + B = 2x - 1$$

The equation above is about two functions; the two sides must be equal for all values of x . Let us substitute $x = 5$ into both sides:

$$\begin{aligned} A(0) + B &= 9 \\ B &= 9 \end{aligned}$$

The other value of x can be arbitrarily chosen. (There is no value that would eliminate B from the equation.) For easy substitution, let us substitute $x = 0$ into both sides and also substitute $B = 9$:

$$\begin{aligned} A(-5) + 9 &= -1 \\ -5A &= -10 \\ A &= 2 \end{aligned}$$

and so $A = 2$ and $B = 9$.

$$7. \int \frac{x+3}{(x-1)^3} dx$$

Solution: We re-write the fraction $\frac{x+3}{(x-1)^3}$ as a sum (or difference) of fractions with denominators $x-1$, $(x-1)^2$ and $(x-1)^3$. This means that we need to solve for A , B , and C in the equation

$$\frac{A}{x-1} + \frac{B}{(x-1)^2} + \frac{C}{(x-1)^3} = \frac{x+3}{(x-1)^3}$$

To simplify the left-hand side, we bring the fractions to the common denominator:

$$\begin{aligned} \frac{A}{x-1} + \frac{B}{(x-1)^2} + \frac{C}{(x-1)^3} &= \frac{A(x-1)^2}{(x-1)^3} + \frac{B(x-1)}{(x-1)^3} + \frac{C}{(x-1)^3} = \frac{A(x-1)^2 + B(x-1) + C}{(x-1)^3} \\ &= \frac{A(x^2 - 2x + 1) + B(x-1) + C}{(x-1)^3} = \frac{Ax^2 - 2Ax + A + Bx - B + C}{(x-1)^3} \\ &= \frac{Ax^2 + (-2A + B)x + A - B + C}{(x-1)^3} \end{aligned}$$

Thus

$$\frac{Ax^2 + (-2A + B)x + A - B + C}{(x-1)^3} = \frac{x+3}{(x-1)^3}$$

We clear the denominators by multiplication

$$Ax^2 + (-2A + B)x + A - B + C = x + 3$$

The equation above is about two polynomials: they are equal to each other as functions and so they must be identical, coefficient by coefficient. We have an equation for each coefficient that gives us a system of linear equations:

$$\begin{aligned} A &= 0 \\ -2A + B &= 1 \\ A - B + C &= 3 \end{aligned}$$

Since $A = 0$, this is really a system in two variables:

$$\begin{aligned} B &= 1 \\ -B + C &= 3 \end{aligned}$$

We solve this system and obtain $B = 1$ and $C = 4$.

So our fraction, $\frac{x+3}{(x-1)^3}$ can be re-written as $\frac{1}{(x-1)^2} + \frac{4}{(x-1)^3}$. We check:

$$\frac{1}{(x-1)^2} + \frac{4}{(x-1)^3} = \frac{1(x-1)}{(x-1)^3} + \frac{4}{(x-1)^3} = \frac{x-1+4}{(x-1)^3} = \frac{x+3}{(x-1)^3}$$

Now we can easily integrate:

$$\begin{aligned} \int \frac{x+3}{(x-1)^3} dx &= \int \frac{1}{(x-1)^2} + \frac{4}{(x-1)^3} dx = \int \frac{1}{(x-1)^2} dx + 4 \int \frac{1}{(x-1)^3} dx \\ &= -\frac{1}{x-1} - \frac{4}{2} \cdot \frac{1}{(x-1)^2} + C = \boxed{-\frac{1}{x-1} - \frac{2}{(x-1)^2} + C} \\ &= \frac{-1(x-1)}{(x-1)^2} - \frac{2}{(x-1)^2} + C = \frac{-x+1-2}{(x-1)^2} + C = \boxed{\frac{-x-1}{(x-1)^2} + C} \end{aligned}$$

Both final answers are acceptable.

Method 2: The values of A , B , and C can be found using a slightly different method as follows. Consider first the equation

$$\frac{A}{x-1} + \frac{B}{(x-1)^2} + \frac{C}{(x-1)^3} = \frac{x+3}{(x-1)^3}$$

We bring the fractions to the common denominator:

$$\frac{A(x-1)^2}{(x-1)^3} + \frac{B(x-1)}{(x-1)^3} + \frac{C}{(x-1)^3} = \frac{x+3}{(x-1)^3}$$

and then multiply both sides by the denominator:

$$A(x-1)^2 + B(x-1) + C = x+3$$

The equation above is about two functions; the two sides must be equal for all values of x . Let us substitute $x = 1$ into both sides:

$$\begin{aligned} A(0) + B(0) + C &= 1+3 \\ C &= 4 \end{aligned}$$

There is no value other than 1 that would eliminate A or B from the equation. Our method will still work. For easy substitution, let us substitute $x = 0$ into both sides and also substitute $C = 4$:

$$\begin{aligned} A(x-1)^2 + B(x-1) + C &= x+3 \\ A(0-1)^2 + B(0-1) + 4 &= 0+3 \\ A - B + 4 &= 3 \\ A - B &= -1 \end{aligned}$$

Let us substitute $x = 2$ into both sides:

$$\begin{aligned} A(x-1)^2 + B(x-1) + C &= x+3 \\ A(2-1)^2 + B(2-1) + 4 &= 2+3 \\ A+B+4 &= 5 \\ A+B &= 1 \end{aligned}$$

We now solve the system of equations

$$\begin{aligned} A-B &= -1 \\ A+B &= 1 \end{aligned}$$

and obtain $A = 0$ and $B = 1$. Recall that we already have $C = 4$.

8. $\int \frac{x^4}{x^4-1} dx$

Solution: This rational function is an improper fraction since the numerator has the same degree as the denominator. We first perform long division. This one is an easy one; the method featured below is called smuggling.

$$\frac{x^4}{x^4-1} = \frac{x^4-1+1}{x^4-1} = \frac{x^4-1}{x^4-1} + \frac{1}{x^4-1} = 1 + \frac{1}{x^4-1}$$

Thus

$$\int \frac{x^4}{x^4-1} dx = \int 1 + \frac{1}{x^4-1} dx = \int 1 dx + \int \frac{1}{x^4-1} dx = x + C_1 + \int \frac{1}{x^4-1} dx$$

We apply the method of partial fractions to compute $\int \frac{1}{x^4-1} dx$.

We factor the denominator: $x^4-1 = (x^2+1)(x+1)(x-1)$. Next, we re-write the fraction $\frac{1}{x^4-1}$ as a sum (or difference) of fractions with denominators x^2+1 and $x+1$, and $x-1$. In the fraction with quadratic denominator, the numerator is linear. This means that we need to solve for A and B in the equation

$$\frac{Ax+B}{x^2+1} + \frac{C}{x+1} + \frac{D}{x-1} = \frac{1}{x^4-1}$$

To simplify the left-hand side, we bring the fractions to the common denominator:

$$\begin{aligned} \frac{Ax+B}{x^2+1} + \frac{C}{x+1} + \frac{D}{x-1} &= \frac{(Ax+B)(x+1)(x-1)}{(x^2+1)(x+1)(x-1)} + \frac{C(x^2+1)(x-1)}{(x^2+1)(x+1)(x-1)} + \frac{D(x^2+1)(x+1)}{(x^2+1)(x+1)(x-1)} \\ &= \frac{(Ax+B)(x+1)(x-1) + C(x^2+1)(x-1) + D(x^2+1)(x+1)}{x^4-1} \\ &= \frac{(Ax+B)(x^2-1) + C(x^3-x^2+x-1) + D(x^3+x^2+x+1)}{x^4-1} \\ &= \frac{Ax^3+Bx^2-Ax-B+Cx^3-Cx^2+Cx-C+Dx^3+Dx^2+Dx+D}{x^4-1} \\ &= \frac{(A+C+D)x^3 + (B-C+D)x^2 + (-A+C+D)x - B-C+D}{x^4-1} \end{aligned}$$

Thus

$$\frac{(A+C+D)x^3 + (B-C+D)x^2 + (-A+C+D)x - B-C+D}{x^4-1} = \frac{1}{x^4-1}$$

We clear the denominators by multiplication

$$(A + C + D)x^3 + (B - C + D)x^2 + (-A + C + D)x - B - C + D = 1$$

The equation above is about two polynomials: they are equal to each other as functions and so they must be identical, coefficient by coefficient. This gives us an equation for each coefficient that forms a system of linear equations:

$$\begin{array}{rrrrrrcl} A & & & + & C & + & D & = & 0 \\ & B & - & C & + & D & & = & 0 \\ -A & & & + & C & + & D & = & 0 \\ & - & B & - & C & + & D & = & 1 \end{array}$$

We will solve this system by elimination. First, we will eliminate A using the first equation. The second and fourth equations do not have A in them, so there is nothing to do there. To eliminate A from the third equation, we add the first one to it.

$$\begin{array}{rrrrrrcl} B & - & C & + & D & = & 0 \\ & & 2C & + & 2D & = & 0 \\ - & B & - & C & + & D & = & 1 \end{array}$$

We now have three equations with three unknowns. We will use the first equation to eliminate B . In case of the second equation, again, there is nothing to do. We add the first equation to the third one to eliminate B .

$$\begin{array}{rrcl} 2C & + & 2D & = & 0 \\ -2C & + & 2D & = & 1 \end{array}$$

Adding the two equations eliminates C and gives us $4D = 1$ and so $D = \frac{1}{4}$. Next, we compute C using the equation

$$\begin{array}{rrcl} 2C + 2D & = & 0 \\ 2C + 2\left(\frac{1}{4}\right) & = & 0 \\ 2C & = & -\frac{1}{2} \\ C & = & -\frac{1}{4} \end{array}$$

We can compute A using the first equation, $A + C + D = 0$

$$\begin{array}{rrcl} A + C + D & = & 0 \\ A - \frac{1}{4} + \frac{1}{4} & = & 0 \\ A & = & 0 \end{array}$$

and we can compute B using the second equation,

$$\begin{array}{rrcl} B - C + D & = & 0 \\ B - \left(-\frac{1}{4}\right) + \frac{1}{4} & = & 0 \\ B & = & -\frac{1}{2} \end{array}$$

Thus $A = 0$, $B = -\frac{1}{2}$, $C = -\frac{1}{4}$, and $D = \frac{1}{4}$.

So our fraction, $\frac{1}{x^4-1}$ can be re-written as $\frac{-\frac{1}{2}}{x^2+1} - \frac{\frac{1}{4}}{x+1} + \frac{\frac{1}{4}}{x-1}$. We check:

$$\begin{aligned} \frac{-\frac{1}{2}}{x^2+1} - \frac{\frac{1}{4}}{x+1} + \frac{\frac{1}{4}}{x-1} &= \frac{-\frac{1}{2}(x+1)(x-1)}{(x^2+1)(x+1)(x-1)} - \frac{\frac{1}{4}(x^2+1)(x-1)}{(x^2+1)(x+1)(x-1)} + \frac{\frac{1}{4}(x^2+1)(x+1)}{(x^2+1)(x+1)(x-1)} \\ &= \frac{-\frac{1}{2}(x+1)(x-1) - \frac{1}{4}(x^2+1)(x-1) + \frac{1}{4}(x^2+1)(x+1)}{(x^2+1)(x+1)(x-1)} \\ &= \frac{-\frac{1}{2}(x^2-1) - \frac{1}{4}(x^3-x^2+x-1) + \frac{1}{4}(x^3+x^2+x+1)}{x^4-1} \\ &= \frac{-\frac{1}{2}x^2 + \frac{1}{2} - \frac{1}{4}x^3 + \frac{1}{4}x^2 - \frac{1}{4}x + \frac{1}{4} + \frac{1}{4}x^3 + \frac{1}{4}x^2 + \frac{1}{4}x + \frac{1}{4}}{x^4-1} \\ &= \frac{\left(-\frac{1}{4} + \frac{1}{4}\right)x^3 + \left(-\frac{1}{2} + \frac{1}{4} + \frac{1}{4}\right)x^2 + \left(-\frac{1}{4} + \frac{1}{4}\right)x + \frac{1}{2} + \frac{1}{4} + \frac{1}{4}}{x^4-1} = \frac{1}{x^4-1} \end{aligned}$$

Now we re-write the integral:

$$\begin{aligned} \int \frac{1}{x^4-1} dx &= \int \frac{-\frac{1}{2}}{x^2+1} - \frac{\frac{1}{4}}{x+1} + \frac{\frac{1}{4}}{x-1} dx = -\frac{1}{2} \int \frac{1}{x^2+1} dx - \frac{1}{4} \int \frac{1}{x+1} dx + \frac{1}{4} \int \frac{1}{x-1} dx \\ &= -\frac{1}{2} \arctan x - \frac{1}{4} \ln|x+1| + \frac{1}{4} \ln|x-1| + C \end{aligned}$$

Thus the final answer is

$$\begin{aligned} \int \frac{x^4}{x^4-1} dx &= \int 1 + \frac{x^4}{x^4-1} dx = \int 1 dx + \int \frac{1}{x^4-1} dx \\ &= x + C_1 - \frac{1}{2} \arctan x - \frac{1}{4} \ln|x+1| + \frac{1}{4} \ln|x-1| + C_2 \\ &= \boxed{x - \frac{1}{2} \arctan x - \frac{1}{4} \ln|x+1| + \frac{1}{4} \ln|x-1| + C} \end{aligned}$$

Method 2: The values of A , B , C , and D can be found using a slightly different method as follows. Consider first the equation

$$\frac{Ax+B}{x^2+1} + \frac{C}{x+1} + \frac{D}{x-1} = \frac{1}{x^4-1}$$

We bring the fractions to the common denominator:

$$\frac{(Ax+B)(x+1)(x-1)}{(x^2+1)(x+1)(x-1)} + \frac{C(x^2+1)(x-1)}{(x^2+1)(x+1)(x-1)} + \frac{D(x^2+1)(x+1)}{(x^2+1)(x+1)(x-1)} = \frac{1}{(x^2+1)(x+1)(x-1)}$$

and then multiply both sides by the denominator:

$$(Ax+B)(x+1)(x-1) + C(x^2+1)(x-1) + D(x^2+1)(x+1) = 1$$

The equation above is about two functions; the two sides must be equal for all values of x . Let us substitute $x = -1$ into both sides:

$$\begin{aligned}
 (Ax + B)(x + 1)(x - 1) + C(x^2 + 1)(x - 1) + D(x^2 + 1)(x + 1) &= 1 \\
 (A(-1) + B)((-1) + 1)((-1) - 1) + C((-1)^2 + 1)((-1) - 1) + D((-1)^2 + 1)((-1) + 1) &= 1 \\
 (-A + B)(0)(-2) + C(2)(-2) + D(2)(0) &= 1 \\
 -4C &= 1 \\
 C &= -\frac{1}{4}
 \end{aligned}$$

Let us substitute $x = 1$ into both sides:

$$\begin{aligned}
 (Ax + B)(x + 1)(x - 1) + C(x^2 + 1)(x - 1) + D(x^2 + 1)(x + 1) &= 1 \\
 (A(1) + B)((1) + 1)(1 - 1) + C(1^2 + 1)(1 - 1) + D(1^2 + 1)(1 + 1) &= 1 \\
 (A + B)(2)(0) + C(2)(0) + D(2)(2) &= 1 \\
 4D &= 1 \\
 D &= \frac{1}{4}
 \end{aligned}$$

Let us substitute $x = 0$ into both sides and also $C = -\frac{1}{4}$ and $D = \frac{1}{4}$:

$$\begin{aligned}
 (Ax + B)(x + 1)(x - 1) + C(x^2 + 1)(x - 1) + D(x^2 + 1)(x + 1) &= 1 \\
 (A(0) + B)((0) + 1)(0 - 1) + C(0^2 + 1)(0 - 1) + D(0^2 + 1)(0 + 1) &= 1 \\
 (B)(1)(-1) + C(1)(-1) + D(1)(1) &= 1 \\
 -B - C + D &= 1 \\
 -B - \left(-\frac{1}{4}\right) + \frac{1}{4} &= 1 \\
 -B + \frac{1}{2} &= 1 \\
 -B &= \frac{1}{2} \\
 B &= -\frac{1}{2}
 \end{aligned}$$

Let us substitute $x = 2$ into both sides and also $B = -\frac{1}{2}$, $C = -\frac{1}{4}$ and $D = \frac{1}{4}$:

$$\begin{aligned}
 (Ax + B)(x + 1)(x - 1) + C(x^2 + 1)(x - 1) + D(x^2 + 1)(x + 1) &= 1 \\
 (A(2) + B)((2) + 1)(2 - 1) + C(2^2 + 1)(2 - 1) + D(2^2 + 1)(2 + 1) &= 1 \\
 (2A + B)(3)(1) + C(5)(1) + D(5)(3) &= 1 \\
 3(2A + B) + 5C + 15D &= 1 \\
 6A + 3B + 5C + 15D &= 1 \\
 6A + 3\left(-\frac{1}{2}\right) + 5\left(-\frac{1}{4}\right) + 15\left(\frac{1}{4}\right) &= 1 \\
 6A - \frac{3}{2} - \frac{5}{4} + \frac{15}{4} &= 1 \\
 6A + \frac{-6 - 5 + 15}{4} &= 1
 \end{aligned}$$

$$\begin{aligned}
6A + \frac{4}{4} &= 1 \\
6A + 1 &= 1 \\
6A &= 0 \\
A &= 0
\end{aligned}$$

and so $A = 0$, $B = -\frac{1}{2}$, $C = -\frac{1}{4}$, and $D = \frac{1}{4}$.

$$9. \int \sec x \, dx = \frac{1}{2} \ln \left| \frac{1 + \sin x}{1 - \sin x} \right| + C = \ln |\sec x + \tan x| + C = -\ln |\sec x - \tan x| + C$$

Solution:

$$\int \sec x \, dx = \int \frac{1}{\cos x} \, dx = \int \frac{1}{\cos x} \cdot \frac{\cos x}{\cos x} \, dx = \int \frac{\cos x}{\cos^2 x} \, dx = \int \frac{\cos x}{1 - \sin^2 x} \, dx$$

Now let $u = \sin x$. Then $du = \cos x \, dx$.

$$\int \frac{\cos x}{1 - \sin^2 x} \, dx = \int \frac{1}{1 - \sin^2 x} \cos x \, dx = \int \frac{1}{1 - u^2} \, du = \int \frac{1}{(1 - u)(1 + u)} \, du$$

This integral can be computed by partial fractions:

$$\frac{A}{1 - u} + \frac{B}{1 + u} = \frac{1}{1 - u^2}$$

The left-hand side can be re-written

$$\frac{A}{1 - u} + \frac{B}{1 + u} = \frac{A(1 + u)}{(1 - u)(1 + u)} + \frac{B(1 - u)}{(1 + u)(1 - u)} = \frac{Au + A - Bu + B}{1 - u^2} = \frac{(A - B)u + A + B}{1 - u^2}$$

So we have

$$\frac{(A - B)u + A + B}{1 - u^2} = \frac{1}{1 - u^2}$$

We clear the denominators by multiplication

$$(A - B)u + A + B = 1$$

The equation above is about two polynomials: they are equal to each other as functions and so they must be identical, coefficient by coefficient. This gives us an equation for each coefficient that forms a system of linear equations:

$$\begin{aligned}
A - B &= 0 \\
A + B &= 1
\end{aligned}$$

we solve this system and obtain $A = B = \frac{1}{2}$. Indeed,

$$\frac{1}{2} \left(\frac{1}{1 + u} + \frac{1}{1 - u} \right) = \frac{1}{2} \frac{1 - u + 1 + u}{(1 - u)(1 + u)} = \frac{1}{2} \frac{2}{1 - u^2} = \frac{1}{1 - u^2}$$

Now for the integral:

$$\begin{aligned}
 \int \frac{1}{1-u^2} du &= \int \frac{1}{(1-u)(1+u)} du = \int \frac{1}{2} \left(\frac{1}{1-u} + \frac{1}{1+u} \right) du = \frac{1}{2} \int \frac{1}{1-u} du + \frac{1}{2} \int \frac{1}{1+u} du \\
 &= -\frac{1}{2} \ln |1-u| + \frac{1}{2} \ln |1+u| + C = \frac{1}{2} \ln |1+u| - \frac{1}{2} \ln |1-u| + C \\
 &= \boxed{\frac{1}{2} \ln |1 + \sin x| - \frac{1}{2} \ln |1 - \sin x| + C}
 \end{aligned}$$

Note the $-$ sign in $\int \frac{1}{1-u} du = -\ln |1-u| + C$ is caused by the chain rule. Also note that the final answer can be re-written in several forms:

$$\begin{aligned}
 &\frac{1}{2} \ln |1 + \sin x| - \frac{1}{2} \ln |1 - \sin x| = \\
 &= \frac{1}{2} \ln \left| \frac{1 + \sin x}{1 - \sin x} \right| = \frac{1}{2} \ln \left| \frac{1 + \sin x}{1 - \sin x} \cdot \frac{1 + \sin x}{1 + \sin x} \right| = \frac{1}{2} \ln \left| \frac{(1 + \sin x)^2}{1 - \sin^2 x} \right| = \frac{1}{2} \ln \left| \frac{(1 + \sin x)^2}{\cos^2 x} \right| \\
 &= \ln \left| \left(\frac{(1 + \sin x)^2}{\cos^2 x} \right)^{1/2} \right| = \ln \sqrt{\frac{(1 + \sin x)^2}{\cos^2 x}} = \ln \left| \frac{1 + \sin x}{\cos x} \right| = \ln \left| \frac{1}{\cos x} + \frac{\sin x}{\cos x} \right| = \ln |\sec x + \tan x|
 \end{aligned}$$

So, our result can also be presented as $\boxed{\ln |\sec x + \tan x| + C}$

Another form can be obtained as shown below.

$$\begin{aligned}
 &\frac{1}{2} \ln |1 + \sin x| - \frac{1}{2} \ln |1 - \sin x| = \\
 &= \frac{1}{2} \ln \left| \frac{1 + \sin x}{1 - \sin x} \right| = \frac{1}{2} \ln \left| \frac{1 + \sin x}{1 - \sin x} \cdot \frac{1 - \sin x}{1 - \sin x} \right| = \frac{1}{2} \ln \left| \frac{1 - \sin^2 x}{(1 - \sin x)^2} \right| = \frac{1}{2} \ln \left| \frac{\cos^2 x}{(1 - \sin x)^2} \right| \\
 &= \ln \left| \left(\frac{\cos^2 x}{(1 - \sin x)^2} \right)^{1/2} \right| = \ln \sqrt{\frac{\cos^2 x}{(1 - \sin x)^2}} = \ln \left| \frac{\cos x}{1 - \sin x} \right| = \ln \left| \frac{1}{\frac{1 - \sin x}{\cos x}} \right| \\
 &= \ln \left| \left(\frac{1 - \sin x}{\cos x} \right)^{-1} \right| = \ln \left| \frac{1}{\cos x} - \frac{\sin x}{\cos x} \right|^{-1} = -\ln |\sec x - \tan x|
 \end{aligned}$$

So, our result can also be presented as $\boxed{-\ln |\sec x - \tan x| + C}$. Actually, there are more forms possible, but we will stop here.

10. $\int \operatorname{csch} x \, dx$

Solution: This is an interesting application of partial fractions.

$$\int \operatorname{csch} x \, dx = \int \frac{2}{e^x - e^{-x}} \, dx = \int \frac{2}{e^x - \frac{1}{e^x}} \, dx$$

we now multiply both numerator and denominator by e^x .

$$\int \frac{2}{e^x - \frac{1}{e^x}} dx = \int \frac{2e^x}{(e^x)^2 - 1} dx$$

We proceed with a substitution: let $u = e^x$. Then $du = e^x dx$ and so

$$\int \frac{2}{(e^x)^2 - 1} (e^x dx) = \int \frac{2}{u^2 - 1} du$$

This is now an integral we can easily compute via partial fractions. We easily decompose $\frac{2}{u^2 - 1}$ as

$$\frac{1}{u - 1} - \frac{1}{u + 1}$$

$$\begin{aligned} \int \operatorname{csch} x \, dx &= \int \frac{2}{u^2 - 1} du = \int \left(\frac{1}{u - 1} - \frac{1}{u + 1} \right) du = \int \frac{1}{u - 1} du - \int \frac{1}{u + 1} du = \ln |u - 1| - \ln |u + 1| + C \\ &= \boxed{\ln |e^x - 1| - \ln (e^x + 1) + C} \end{aligned}$$

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