

1. (Constant rule) If $f(x) = c$ where c is a constant, then $f'(x) = 0$.
2. (Power rule) If $f(x) = x^n$, then $f'(x) = nx^{n-1}$
3. (Constant multiplier rule) For any differentiable function $f(x)$ and constant c , we define the function cf as $(cf)(x) = c \cdot f(x)$. Then $(cf)'(x) = c \cdot f'(x)$
4. (Sum rule) If f and g are differentiable functions, then so is $f + g$, and $(f + g)'(x) = f'(x) + g'(x)$.

Proofs of these theorems

1. (Constant rule) If $f(x) = c$ where c is a constant, then $f'(x) = 0$.

Proof:

$$f'(x) = \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \rightarrow 0} \frac{c - c}{h} = \lim_{h \rightarrow 0} \frac{0}{h} = \lim_{h \rightarrow 0} 0 = 0 \blacksquare$$

2. (Power rule) If $f(x) = x^n$, then $f'(x) = nx^{n-1}$

We will not prove this general statement here, although it is true for every real number n . Instead, we will prove it for $n = 2$, $n = 3$, $n = -1$, and $n = \frac{1}{2}$

2A. Claim: If $f(x) = x^2$, then $f'(x) = 2x$

Proof:

$$\begin{aligned} f'(x) &= \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \rightarrow 0} \frac{(x+h)^2 - x^2}{h} = \lim_{h \rightarrow 0} \frac{x^2 + 2xh + h^2 - x^2}{h} = \lim_{h \rightarrow 0} \frac{2xh + h^2}{h} \\ &= \lim_{h \rightarrow 0} \frac{h(2x+h)}{h} = \lim_{h \rightarrow 0} 2x + h = 2x \blacksquare \end{aligned}$$

2B. Claim: If $f(x) = x^3$, then $f'(x) = 3x^2$

Proof:

$$\begin{aligned} f'(x) &= \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \rightarrow 0} \frac{(x+h)^3 - x^3}{h} = \lim_{h \rightarrow 0} \frac{x^3 + 3x^2h + 3xh^2 + h^3 - x^3}{h} \\ &= \lim_{h \rightarrow 0} \frac{3x^2h + 3xh^2 + h^3}{h} = \lim_{h \rightarrow 0} \frac{h(3x^2 + 3xh + h^2)}{h} = \lim_{h \rightarrow 0} 3x^2 + 3xh + h^2 = 3x^2 \blacksquare \end{aligned}$$

2C. Claim: If $f(x) = \frac{1}{x}$, then $f'(x) = -\frac{1}{x^2}$

Proof:

$$\begin{aligned} f'(x) &= \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \rightarrow 0} \frac{\frac{1}{x+h} - \frac{1}{x}}{h} = \lim_{h \rightarrow 0} \frac{\frac{x}{x(x+h)} - \frac{x+h}{x(x+h)}}{h} = \lim_{h \rightarrow 0} \frac{x - (x+h)}{x(x+h)} \cdot \frac{1}{h} \\ &= \lim_{h \rightarrow 0} \frac{x - x - h}{x(x+h)} \cdot \frac{1}{h} = \lim_{h \rightarrow 0} \frac{-h}{x(x+h)} \cdot \frac{1}{h} = \lim_{h \rightarrow 0} \frac{-1}{x(x+h)} = -\frac{1}{x^2} \blacksquare \end{aligned}$$

2D. Claim: If $f(x) = \sqrt{x}$, then $f'(x) = \frac{1}{2\sqrt{x}}$

Proof:

$$\begin{aligned} f'(x) &= \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \rightarrow 0} \frac{\sqrt{x+h} - \sqrt{x}}{h} = \lim_{h \rightarrow 0} \frac{\sqrt{x+h} - \sqrt{x}}{h} \cdot \frac{\sqrt{x+h} + \sqrt{x}}{\sqrt{x+h} + \sqrt{x}} \\ &= \lim_{h \rightarrow 0} \frac{(\sqrt{x+h} - \sqrt{x})(\sqrt{x+h} + \sqrt{x})}{h(\sqrt{x+h} + \sqrt{x})} = \lim_{h \rightarrow 0} \frac{(\sqrt{x+h})^2 - (\sqrt{x})^2}{h(\sqrt{x+h} + \sqrt{x})} = \lim_{h \rightarrow 0} \frac{x+h-x}{h(\sqrt{x+h} + \sqrt{x})} \\ &= \lim_{h \rightarrow 0} \frac{h}{h(\sqrt{x+h} + \sqrt{x})} = \lim_{h \rightarrow 0} \frac{1}{\sqrt{x+h} + \sqrt{x}} = \frac{1}{2\sqrt{x}} \blacksquare \end{aligned}$$

3. (Constant multiplier rule) For any differentiable function $f(x)$ and constant c , we define the function cf as $(cf)(x) = c \cdot f(x)$. Then $(cf)'(x) = c \cdot f'(x)$

Proof:

$$\begin{aligned} (cf)'(x) &= \lim_{h \rightarrow 0} \frac{(cf)(x+h) - (cf)(x)}{h} = \lim_{h \rightarrow 0} \frac{c \cdot f(x+h) - c \cdot f(x)}{h} = \lim_{h \rightarrow 0} \frac{c(f(x+h) - f(x))}{h} \\ &= c \cdot \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} = c \cdot f'(x) \blacksquare \end{aligned}$$

4. (Sum rule) If f and g are differentiable functions, then so is $f+g$, and $(f+g)'(x) = f'(x) + g'(x)$.

Proof:

$$\begin{aligned} (f+g)'(x) &= \lim_{h \rightarrow 0} \frac{(f+g)(x+h) - (f+g)(x)}{h} = \lim_{h \rightarrow 0} \frac{[f(x+h) + g(x+h)] - [f(x) + g(x)]}{h} \\ &= \lim_{h \rightarrow 0} \frac{f(x+h) + g(x+h) - f(x) - g(x)}{h} = \lim_{h \rightarrow 0} \frac{f(x+h) - f(x) + g(x+h) - g(x)}{h} \\ &= \lim_{h \rightarrow 0} \left(\frac{f(x+h) - f(x)}{h} + \frac{g(x+h) - g(x)}{h} \right) \\ &= \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} + \lim_{h \rightarrow 0} \frac{g(x+h) - g(x)}{h} = f'(x) + g'(x) \blacksquare \end{aligned}$$

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