

The Axioms of Real Numbers

A **definition** is a type of statement in which we agree how we will refer to things. It is true in a sense because it just sets an agreement about labeling things.

An **axiom** is a statement that we accept as true, without requiring proof of it. It usually agrees with our natural instincts and they "feel true". One example is the statement: "Two points uniquely determine a straight line".

A **theorem** is a statement that we prove to be true. But what does it mean to prove something? It means to derive it from the axioms. Mathematicians set down a set of basic 'truths', the axioms. Everything we prove, we derive them from the axioms. All theorems about the real numbers can be derived from the following axioms.

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| <p>A1. Addition is commutative.
For all $x, y \in \mathbb{R}$, $x + y = y + x$.</p> <p>A2. Addition is associative.
For all $x, y, z \in \mathbb{R}$, $(x + y) + z = x + (y + z)$.</p> <p>A3. Additive identity.
There exists a real number d, such that for every real number x, $x + d = x$. (This number is 0).</p> <p>A4. Additive inverse.
For all $x \in \mathbb{R}$, there exists a real number x^*, such that $x + x^* = 0$. (We denote this number by $-x$).</p> | <p>M1. Multiplication is commutative.
For all $x, y \in \mathbb{R}$, $xy = yx$.</p> <p>M2. Multiplication is associative.
For all $x, y, z \in \mathbb{R}$, $(xy)z = x(yz)$.</p> <p>M3. Multiplicative identity.
There exists a real number d, such that for all $x \in \mathbb{R}$, $xd = x$. (This number is 1.)</p> <p>M4. Multiplicative inverse.
For all $x \in \mathbb{R}$, $x \neq 0$, there exists a real number x^*, such that $xx^* = 1$. (We denote this number by $\frac{1}{x}$).</p> |
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- D1. Distributive Law.
For all $x, y, z \in \mathbb{R}$, $z(x + y) = zx + zy$.

Sets that have these properties are called fields, and are studied in depth in abstract algebra.

The real number system also has the following axioms about ordering.

- O1. For all $x, y \in \mathbb{R}$, either $a \leq b$ or $b \leq a$ or both.
- O2. If $a \leq b$ and $b \leq a$, then $a = b$.
- O3. If $a \leq b$ and $b \leq c$, then $a \leq c$.
- O4. If $a \leq b$ then $a + c \leq b + c$.
- O5. If $a \leq b$ and $0 \leq c$, then $ac \leq bc$.

A set with all these properties is called an ordered field. However, so far these properties hold for both $\mathbb{Q}$ (the set of all rational numbers) and $\mathbb{R}$ (the set of all real numbers.) What distinguishes these two is the completeness property, that is true for $\mathbb{R}$ but not for $\mathbb{Q}$.

Completeness property: Every non-empty subset of $\mathbb{R}$ that is bounded above has a least upper bound.

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